

# AN A POSTERIORI ANALYSIS OF SOME INCONSISTENT, NONCONFORMING GALERKIN METHODS APPROXIMATING ELLIPTIC PROBLEMS

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**ABSTRACT.** In this work we present an a posteriori analysis for classes of inconsistent, nonconforming schemes approximating elliptic problems. We show the estimates coincide with existing ones for interior penalty type discontinuous Galerkin approximations of the Laplacian and give new estimates for inconsistent discontinuous Galerkin approximation schemes of elliptic problems under quadrature approximation. We also examine the effect of inconsistencies on the a posteriori analysis of schemes applied to an unbalanced problem.

## 1. INTRODUCTION

The design of efficient numerical approximations of partial differential equations (PDEs) is of paramount importance in scientific computing. To this end a large amount of work has been carried out on finite element methods and, in particular, on the a posteriori analysis of such methods. The a posteriori framework allows for the construction of adaptive methods that, given a fixed number of degrees of freedom, are able to better approximate nonsmooth solutions of PDEs than the nonadaptive counterparts [AO00, Ver96].

Discontinuous Galerkin (dG) methods form a class of finite element approximation schemes. These, as the name suggest, are constructed without enforcing continuity of the discrete solution over the domain. For second order elliptic problems these are nonconforming methods. See [ABCM02] for an accessible overview and history of these methods for second order problems. For higher order problems, for example the (nonlinear) biharmonic problem, dG methods are a useful alternative to using  $C^1$  conforming elements whose derivation and implementation can become very complicated [Bak77, GHV11, Pry14]. The historical origins of the method can be traced back to that of the transport equation and, as such, the dG methods in the non-selfadjoint case, for example convection dominated problems, are extremely powerful [Coc99].

In the elliptic case there are two main approaches to the a posteriori analysis of such nonconforming schemes. The first is based on a Helmholtz decomposition of the error [BHL03] and the second based on reconstruction operators [KP03]. The reconstruction approach is extremely versatile and has been applied not just to elliptic problems but also to those of hyperbolic type [GHM14, GMP15]. In both scenarios the main idea is to develop an appropriate *reconstruction* of the dG object which is itself smooth enough to apply stability arguments of the underlying PDE. The blanket assumption in the elliptic case is that, up to mesh dependent terms, the dG approximation is “close enough” to a conforming approximation which is indeed true for primal methods chosen with large enough penalty parameters [DPE12].

In this work we utilise these reconstruction operators to develop an abstract a posteriori framework for use in elliptic problems. We are particularly interested in the effect of inconsistencies in the estimate. The inconsistencies that we on focus may arise in various forms, quadrature approximation being the most obvious example. This has been studied previously in the form of data oscillation in the convergence of  $h$ -adaptive algorithms [MNS00, c.f.]. Another related work is [NSSV06] where the authors study quadrature effects of which arise from a nonlinear source term. Inconsistencies may come from other sources though, for example, stabilisation introduced in the approximation of convection dominated problems [Bur05] may be viewed in this fashion. Mixed methods like an operator splitting of a nonconstant coefficient fourth order problem or discretisation of a nonvariational problem [LP11, DP13] are inherently inconsistent as are those of virtual element type [BdVBC<sup>+</sup>13]. With this range of inconsistencies in mind we build our a posteriori estimate to resemble Strang’s Lemma, the starting point when conducting a priori error analysis of inconsistent methods. We show that as long as a reconstruction operator is available, inconsistencies such as those mentioned above

are a posteriori controllable. Interestingly, depending on the class of inconsistency, it is not always necessary to a posteriori compute the reconstruction, in some cases it suffices to only have existence of such an operator.

By building the a posteriori bounds in this way and using ingredients from standard a posteriori analysis we can deal with problems as diverse as unbalanced problems like linear nonvariational and nonlinear  $p$ -Laplacian or  $p$ -biharmonic type, although in this introductory work we will restrict our attention to linear problems. Specifically, to illustrate the approach, we initially focus on the comparison of two dG discretisations of the Laplacian which are standard in the literature. We move onto the a posteriori effect that quadrature error has on finite element discretisations of a model problem. We show that the estimator can be split into components of which some terms can be computed *exactly* using the same quadrature approximation in the scheme and others requiring further approximation. We also examine unbalanced elliptic problems that are nonvariational. We derive  $H^2$  a posteriori bounds which are shown to be reliable and efficient up to data oscillation terms. We also examine further effects of quadrature approximation in this setting and compare with the variational case.

The paper is set out as follows: In §2 we look at an abstract model problem and develop an a posteriori Strang type estimate. In §3 we then illustrate how this framework can be utilised for a model problem and consider inconsistencies arising from quadrature approximations. In §4 we study the application of unbalanced elliptic problems, these are nonvariational in nature and we conduct an a posteriori analysis of existing schemes proposed for this class of problem. Finally, in §5 we summarise extensive numerical experiments testing the robustness of the proposed estimators.

## 2. ABSTRACT PROBLEM SETUP

Throughout this section we develop an abstract a posteriori framework for using notation borrowed from [EG04, §2]. To that end let  $W$  be a Banach space and  $V$  a reflexive Banach space, equipped with norms  $\|\cdot\|_W, \|\cdot\|_V$  respectively. Let  $\mathcal{A} : W \times V \rightarrow \mathbb{R}$  be a bilinear form and  $l : V \rightarrow \mathbb{R}$  be a linear form. Consider the abstract problem to find  $u \in W$  such that

$$(2.1) \quad \mathcal{A}(u, v) = l(v) \quad \forall v \in V.$$

**2.1. Assumption** (The continuous problem is inf-sup stable). Assume that this problem is stable, in the sense that for some  $\gamma > 0$

$$(2.2) \quad \sup_{v \in V, \|v\|_V \leq 1} \mathcal{A}(u, v) \geq \gamma \|u\|_W.$$

**2.2. Proposition** (A priori bound). *Under the conditions of Assumption 2.1 we have the following a priori bound for the solution of (2.1):*

$$(2.3) \quad \|u\|_W \leq C \|l\|_{V^*},$$

where  $V^*$  is the dual space of  $V$ .

We let  $V_h, W_h$  be finite dimensional (not necessarily conforming) spaces. Suppose that  $V_h, W_h$  are equipped with norms  $\|\cdot\|_{V_h}, \|\cdot\|_{W_h}$  respectively and  $W_h$  a seminorm  $|\cdot|_{W_h}$ . Let  $W(h) = W + W_h$  and let  $W(h)$  be equipped with the norm  $\|\cdot\|_{W(h)}$  such that

$$(2.4) \quad \|w_h\|_{W(h)} = \|w_h\|_{W_h} \quad \forall w_h \in W_h$$

$$(2.5) \quad \|w\|_{W(h)} = \|w\|_W \quad \forall w \in W$$

$$(2.6) \quad |w|_{W_h} = 0 \quad \forall w \in W$$

$$(2.7) \quad \|v\|_{V_h} = \|v\|_V \quad \forall v \in V.$$

We consider the Galerkin method to seek  $u_h \in W_h$  such that

$$(2.8) \quad \mathcal{A}_h(u_h, v_h) = l_h(v_h) \quad \forall v_h \in V_h,$$

where  $\mathcal{A}_h(\cdot, \cdot) : W_h \times V_h \rightarrow \mathbb{R}$  is an approximation of  $\mathcal{A}(\cdot, \cdot)$  and  $l_h(\cdot) : V_h \rightarrow \mathbb{R}$  is an approximation of  $l(\cdot)$ .

**2.3. Assumption** (On the Galerkin method). We will assume that the Galerkin scheme is bounded in the sense that

$$(2.9) \quad \mathcal{A}_h(w_h, v_h) \leq C_B \|w_h\|_{W_h} \|v_h\|_{V_h} \quad \forall w_h \in W_h, v_h \in V_h.$$

**2.4. Remark.** Note that there is *no* assumption on the convergence of the Galerkin scheme, only that the problem is stable.

**2.5. Assumption** (Conforming finite dimensional space). Suppose another finite dimensional space,  $\mathbb{W}$ , exists and is a conforming approximation of  $W$ , i.e.,  $\mathbb{W} \subset W$  and there exists a *reconstruction operator*  $E : W_h \rightarrow \mathbb{W}$ .

**2.6. Theorem** (Strang type a posteriori result). Let  $e := u - u_h = (u - E(u_h)) + (E(u_h) - u_h) =: e^C + e^N$  then we have the a posteriori result that for any  $v_h \in V_h$

$$\begin{aligned}
\|e\|_{W_h} &\leq \frac{1}{\gamma} \left( \sup_{v \in V, \|v\|_V \leq 1} [l_h(v - v_h) - \mathcal{A}_h(u_h, v - v_h)] + \sup_{v \in V, \|v\|_V \leq 1} [l(v) - l_h(v)] \right. \\
(2.10) \quad &\quad \left. + \sup_{v \in V, \|v\|_V \leq 1} [\mathcal{A}_h(E(u_h), v) - \mathcal{A}(E(u_h), v)] \right) + \left( 1 + \frac{C_B}{\gamma} \right) \|e^N\|_{W_h} \\
&=: \frac{1}{\gamma} (\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3 + (\gamma + C_B) \mathcal{E}_4).
\end{aligned}$$

**2.7. Remark** (Structure of the estimate). It should be noted that the result in Theorem 2.6 is not a posteriori computable since it involves taking the supremum over an infinite dimensional space. It is, however, the beginnings of an a posteriori bound. The first term  $\mathcal{E}_1$  represents a standard residual term which can be computably estimated using standard techniques [AO00, c.f.]. The second and third terms  $\mathcal{E}_2, \mathcal{E}_3$  measure a form of inconsistency error and the fourth term,  $\mathcal{E}_4$ , the nonconformity error.

**2.8. Remark** (Comparison with the a priori Strang Lemma). Recall the Strang a priori result that if the bilinear form associated to the Galerkin method (2.8) satisfies a discrete inf-sup condition, that is

$$(2.11) \quad \sup_{v_h \in V_h, \|v_h\|_{V_h} \leq 1} \mathcal{A}_h(w_h, v_h) \geq \gamma_h \|w_h\|_{W_h}$$

then

$$(2.12) \quad \|e\|_{W_h} \leq \left( 1 + \frac{C_B}{\gamma_h} \right) \inf_{w_h \in V_h} \|u - w_h\|_{W_h} + \frac{1}{\gamma_h} \left( \sup_{v_h \in V_h, \|v_h\|_{V_h} \leq 1} [l_h(v_h) - \mathcal{A}_h(u, v_h)] \right).$$

The inconsistency term represents how badly the solution  $u$  satisfies the discrete scheme. Theorem 2.6 may be reformulated as

$$\begin{aligned}
\|e\|_{W_h} &\leq \frac{1}{\gamma} \left( \sup_{v \in V, \|v\|_V \leq 1} [l_h(v - v_h) - \mathcal{A}_h(u_h, v - v_h)] + \sup_{v \in V, \|v\|_V \leq 1} [l(v) - \mathcal{A}(E(u_h), v)] \right. \\
(2.13) \quad &\quad \left. + \sup_{v \in V, \|v\|_V \leq 1} [l_h(v) - \mathcal{A}_h(E(u_h), v)] \right) + \left( 1 + \frac{C_B}{\gamma} \right) \|e^N\|_{W_h}
\end{aligned}$$

and we may think of the inconsistencies as how badly the reconstruction  $E(u_h)$  satisfies both the smooth problem and the Galerkin approximation.

**Proof** of Theorem 2.6. Using the stability of the continuous problem from Assumption 2.1 we have for any  $v_h \in V_h$

$$\begin{aligned}
\|e^C\|_W &\leq \frac{1}{\gamma} \left( \sup_{v \in V, \|v\|_V \leq 1} \mathcal{A}(e^C, v) \right) = \frac{1}{\gamma} \left( \sup_{v \in V, \|v\|_V \leq 1} [\mathcal{A}(u, v) - \mathcal{A}(E(u_h), v)] \right) \\
(2.14) \quad &\leq \frac{1}{\gamma} \left( \sup_{v \in V, \|v\|_V \leq 1} [l(v) - \mathcal{A}(E(u_h), v)] \right),
\end{aligned}$$

using the continuous problem (2.1). Adding and subtracting appropriate terms, we have

$$\begin{aligned}
\|e^C\|_W &\leq \frac{1}{\gamma} \left( \sup_{v \in V, \|v\|_V \leq 1} \left[ l(v) - l_h(v) + l_h(v - v_h) - \mathcal{A}_h(u_h, v - v_h) \right. \right. \\
&\quad \left. \left. - \mathcal{A}_h(E(u_h) - u_h, v) + \mathcal{A}_h(E(u_h), v) - \mathcal{A}(E(u_h), v) \right] \right) \\
(2.15) \quad &\leq \frac{1}{\gamma} \left( \sup_{v \in V, \|v\|_V \leq 1} [l_h(v - v_h) - \mathcal{A}_h(u_h, v - v_h)] + \sup_{v \in V, \|v\|_V \leq 1} [l(v) - l_h(v)] \right. \\
&\quad \left. + \sup_{v \in V, \|v\|_V \leq 1} [\mathcal{A}_h(E(u_h), v) - \mathcal{A}(E(u_h), v)] + \sup_{v \in V, \|v\|_V \leq 1} \mathcal{A}_h(E(u_h) - u_h, v) \right),
\end{aligned}$$

in view of the triangle inequality. To conclude we note that

$$(2.16) \quad \sup_{v \in V, \|v\|_V \leq 1} \mathcal{A}_h(E(u_h) - u_h, v) \leq C_B \|e^N\|_{W_h} \|v\|_{V_h},$$

from Assumption 2.3, (2.5) and (2.7). The result then follows from

$$(2.17) \quad \|e\|_{W_h} \leq \|e^C\|_W + \|e^N\|_{W_h},$$

concluding the proof.  $\square$

**2.9. Remark** (Computability of (4.12)). For (4.12) to be useful some assumptions need to be made on the reconstruction  $E$ . In particular, we impose that

$$(2.18) \quad \|E(w_h) - w_h\|_{W_h} \leq C |w_h|_{W_h} \quad \forall w_h \in W_h.$$

Examples of reconstruction operators satisfying this condition for various test problems will be given in subsequent chapters.

**2.10. Definition** (Formal dual problem). Let  $L$  be a Hilbert space such that for any  $v \in V$  we have

$$(2.19) \quad \|v\|_L \leq C \|v\|_V.$$

We define the formal dual (adjoint) problem to (2.1) to be: For any  $g \in L$ , find  $z(g) \in Z \subset V$  such that

$$(2.20) \quad \mathcal{A}^*(z(g), v) = f_L(g, v) \quad \forall v \in V,$$

where  $\mathcal{A}^*(v, w) = \mathcal{A}(w, v)$ .

In addition we suppose that the bilinear form  $f_L(\cdot, \cdot)$  induces a seminorm on  $L$ , i.e.,

$$(2.21) \quad \|\cdot\|_L := \sup_{g \in L} \frac{f_L(g, \cdot)}{\|g\|_V}.$$

**2.11. Assumption.** In addition to Assumption 2.1, assume the formal dual problem is well posed, that is, for all  $g \in L$

$$(2.22) \quad \gamma^* \|z(g)\|_Z \leq \sup_{v \in V} \frac{\mathcal{A}^*(z(g), v)}{\|v\|_V},$$

then we immediately have that

$$(2.23) \quad \|z(g)\|_Z \leq \frac{1}{\gamma^*} \|g\|_L.$$

**2.12. Theorem** (Dual a posteriori bound). *Using the same notation as in Theorem 2.6 under Assumptions 2.1 and 2.11*

$$\begin{aligned}
\|e\|_L &\leq \frac{1}{\gamma^*} \left( \sup_{g \in L, \|g\|_L \leq 1} [l_h(z(g) - z_h) - \mathcal{A}_h(u_h, z(g) - z_h)] + \sup_{g \in L, \|g\|_L \leq 1} [l(z(g)) - l_h(z(g))] \right. \\
(2.24) \quad &\quad \left. + \sup_{g \in L, \|g\|_L \leq 1} [\mathcal{A}_h(E(u_h), z(g)) - \mathcal{A}(E(u_h), z(g))] + \sup_{g \in L, \|g\|_L \leq 1} \mathcal{A}_h(E(u_h) - u_h, z(g)) \right) \\
&=: \frac{1}{\gamma^*} (\mathcal{E}_1^L + \mathcal{E}_2^L + \mathcal{E}_3^L + \mathcal{E}_4^L).
\end{aligned}$$

**Proof** Proceeding along the same lines as Theorem 2.6 we have that  $e := u - u_h = (u - E(u_h)) + (E(u_h) - u_h) =: e^C + e^N$  and using the stability of the dual problem

$$(2.25) \quad \|e^C\|_L = \sup_{g \in L, |g|_L \leq 1} f_L(e^C, g) = \sup_{g \in L, |g|_L \leq 1} \mathcal{A}^*(z(g), e^C) = \sup_{g \in L, |g|_L \leq 1} \mathcal{A}(e^C, z(g)).$$

Adding and subtracting appropriate terms gives

$$(2.26) \quad \begin{aligned} \|e^C\|_L &= \sup_{g \in L, |g|_L \leq 1} l(z(g)) - l_h(z(g)) + l_h(z(g) - z_h) - \mathcal{A}_h(u_h, z(g) - z_h) \\ &\quad - \mathcal{A}_h(E(u_h) - u_h, z(g)) + \mathcal{A}_h(E(u_h), z(g)) - \mathcal{A}(E(u_h), z(g)) \\ &\leq \sup_{g \in L, |g|_L \leq 1} \left[ l_h(z(g) - z_h) - \mathcal{A}_h(u_h, z(g) - z_h) \right] + \sup_{g \in L, |g|_L \leq 1} \left[ l(z(g)) - l_h(z(g)) \right] \\ &\quad + \sup_{g \in L, |g|_L \leq 1} \left[ \mathcal{A}_h(E(u_h), z(g)) - \mathcal{A}(E(u_h), z(g)) \right] + \sup_{g \in L, |g|_L \leq 1} \mathcal{A}_h(E(u_h) - u_h, z(g)), \end{aligned}$$

concluding the proof.  $\square$

**2.13. Remark** (Computability of (2.24)). As in the primal case Theorem 2.12 is not useful in its own right without further assumptions, for example one would desire that the finite element space satisfies some approximability assumption for any  $v \in Z$

$$(2.27) \quad \inf_{v_h \in V_h} \|v - v_h\|_{V_h} \leq C_1 h^\alpha \|v\|_Z.$$

The full range of assumptions required for dual a posteriori error control tends to be very problem specific as will be illustrated in the sequel.

### 3. APPLICATIONS TO DISCONTINUOUS GALERKIN APPROXIMATIONS OF 2ND ORDER LINEAR SELF ADJOINT PROBLEMS

In this section we give an illustrative example showing how to apply the framework to derive a posteriori bounds for various schemes approximating a model problem. To that end, suppose that the function spaces defined in the previous chapter are defined over a domain  $\Omega$ . Let  $\mathcal{T}$  be a conforming, shape regular triangulation of  $\Omega$ , namely,  $\mathcal{T}$  is a finite family of sets such that

- (1)  $K \in \mathcal{T}$  implies  $K$  is an open simplex (segment for  $d = 1$ , triangle for  $d = 2$ , tetrahedron for  $d = 3$ ),
- (2) for any  $K, J \in \mathcal{T}$  we have that  $\overline{K} \cap \overline{J}$  is a full subsimplex (i.e., it is either  $\emptyset$ , a vertex, an edge, a face, or the whole of  $\overline{K}$  and  $\overline{J}$ ) of both  $\overline{K}$  and  $\overline{J}$  and
- (3)  $\bigcup_{K \in \mathcal{T}} \overline{K} = \overline{\Omega}$ .

The shape regularity of  $\mathcal{T}$  is defined as the number

$$(3.1) \quad \mu(\mathcal{T}) := \inf_{K \in \mathcal{T}} \frac{\rho_K}{h_K},$$

where  $\rho_K$  is the radius of the largest ball contained inside  $K$  and  $h_K$  is the diameter of  $K$ . An indexed family of triangulations  $\{\mathcal{T}^n\}_n$  is called *shape regular* if

$$(3.2) \quad \mu := \inf_n \mu(\mathcal{T}^n) > 0.$$

We use the convention where  $h : \Omega \rightarrow \mathbb{R}$  denotes the piecewise constant *meshsize function* of  $\mathcal{T}$ , i.e.,

$$(3.3) \quad h(\mathbf{x}) := \max_{K \ni \mathbf{x}} h_K,$$

which we shall commonly refer to as  $h$ .

We let  $\mathcal{E}$  be the skeleton (set of common interfaces) of the triangulation  $\mathcal{T}$  and say  $e \in \mathcal{E}$  if  $e$  is on the interior of  $\Omega$  and  $e \in \partial\Omega$  if  $e$  lies on the boundary  $\partial\Omega$  and set  $h_e$  to be the diameter of  $e$ .

We let  $\mathbb{P}^k(\mathcal{T})$  denote the space of piecewise polynomials of degree  $k$  over the triangulation  $\mathcal{T}$ , i.e.,

$$(3.4) \quad \mathbb{P}^k(\mathcal{T}) = \{\phi \text{ such that } \phi|_K \in \mathbb{P}^k(K)\}$$

and introduce the *finite element space*

$$(3.5) \quad \mathbb{V} := \mathbb{V}_D(k) = \mathbb{P}^k(\mathcal{T})$$

to be the usual space of discontinuous piecewise polynomial functions of degree  $k$ .

**3.1. Definition** (broken Sobolev spaces, trace spaces). We introduce the broken Sobolev space

$$(3.6) \quad H^l(\mathcal{T}) := \left\{ \phi : \phi|_K \in H^l(K), \text{ for each } K \in \mathcal{T} \right\}.$$

We also make use of functions defined in these broken spaces restricted to the skeleton of the triangulation. This requires an appropriate trace space

$$(3.7) \quad \mathcal{T}(\mathcal{E}) := \prod_{K \in \mathcal{T}} L_2(\partial K).$$

**3.2. Definition** (jumps, averages and tensor jumps). We may define average, jump and tensor jump operators over  $\mathcal{T}(\mathcal{E})$  for arbitrary scalar functions  $v \in \mathcal{T}(\mathcal{E})$  and vectors  $\mathbf{v} \in \mathcal{T}(\mathcal{E})^d$ .

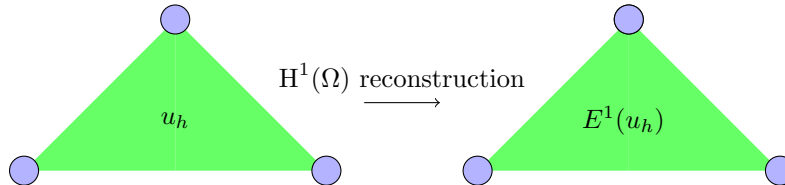
$$\begin{aligned} \{\!\!\{ \cdot \}\!\!\} : \mathcal{T}(\mathcal{E}) &\rightarrow L_2(\mathcal{E}) & \{\!\!\{ \cdot \}\!\!\} : [\mathcal{T}(\mathcal{E})]^d &\rightarrow [L_2(\mathcal{E} \cup \partial\Omega)]^d \\ v &\mapsto \begin{cases} \frac{1}{2}(v|_{K_1} + v|_{K_2}) \\ v|_{\partial\Omega} \text{ on } \partial\Omega \end{cases} & \mathbf{v} &\mapsto \begin{cases} \frac{1}{2}(\mathbf{v}|_{K_1} + \mathbf{v}|_{K_2}) \\ \mathbf{v}|_{\partial\Omega} \text{ on } \partial\Omega \end{cases} \\ \llbracket \cdot \rrbracket : \mathcal{T}(\mathcal{E}) &\rightarrow [L_2(\mathcal{E})]^d & \llbracket \cdot \rrbracket : [\mathcal{T}(\mathcal{E})]^d &\rightarrow L_2(\mathcal{E}) \\ v &\mapsto \begin{cases} v|_{K_1} \mathbf{n}_{K_1} + v|_{K_2} \mathbf{n}_{K_2} \\ (v\mathbf{n})|_{\partial\Omega} \text{ on } \partial\Omega \end{cases} & \mathbf{v} &\mapsto \begin{cases} (\mathbf{v}|_{K_1})^\top \mathbf{n}_{K_1} + (\mathbf{v}|_{K_2})^\top \mathbf{n}_{K_2} \\ (\mathbf{v}^\top \mathbf{n})|_{\partial\Omega} \text{ on } \partial\Omega \end{cases} \\ \llbracket \cdot \rrbracket_\otimes : [\mathcal{T}(\mathcal{E} \cup \partial\Omega)]^d &\rightarrow [L_2(\mathcal{E} \cup \partial\Omega)]^{d \times d} & & \\ \mathbf{v} &\mapsto \begin{cases} \mathbf{v}|_{K_1} \otimes \mathbf{n}_{K_1} + \mathbf{v}|_{K_2} \otimes \mathbf{n}_{K_2} \\ (\mathbf{v} \otimes \mathbf{n})|_{\partial\Omega} \text{ on } \partial\Omega \end{cases} & & \end{aligned}$$

**3.3. Conforming reconstruction operators.** A simple, quite general methodology for the construction of reconstruction operators with the desirable properties mentioned in Remark 2.9 is to use an averaging interpolation operator into an  $H^s$  ( $s = 1, 2$ ) conforming finite element space. For example the Oswald interpolant into a  $C^0$  Lagrangian finite element space can be used for a  $H^1$  conforming operator [KP03] or a  $C^1$  Hsieh–Clough–Tocher macro element conforming space for  $H^2$  conformity [BGS10, GHV11, c.f.].

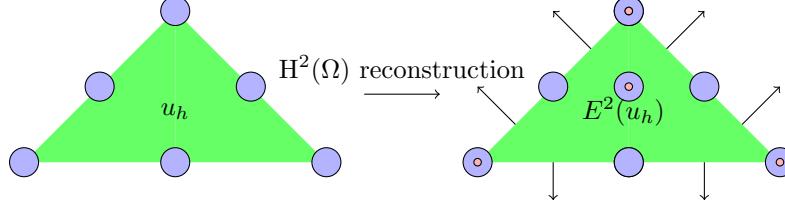
**3.4. Example** (Lowest order  $H^1(\Omega)$  and  $H^2(\Omega)$  reconstructions). An example of the lowest order ( $p = 1$ )  $H^1(\Omega)$  reconstruction operator  $E^1(u_h)$  is for given  $u_h \in \mathbb{V}$  to specify the values of the reconstruction at the vertices as an average in a local neighbourhood, as with the Oswald interpolant [EG04]. In general let  $\mathbf{x}$  be a degree of freedom of the conformal space  $\mathbb{W}$  and let  $\widehat{K_{\mathbf{x}}}$  be the set of all elements sharing the degree of freedom  $\mathbf{x}$  then the reconstruction at that specific degree of freedom is given by

$$(3.8) \quad E(u_h)(\mathbf{x}) = \frac{1}{\text{card}(\widehat{K_{\mathbf{x}}})} \sum_{K \in \widehat{K_{\mathbf{x}}}} u_h|_K(\mathbf{x}).$$

In the case  $k = 1$  the associated degrees of freedom are illustrated below:



Notice that the degrees of freedom of the reconstruction match that of the original function, as such  $E^1(u_h) \in \mathbb{V}$ , this is because of the existence of a conforming  $H^1(\Omega)$  subspace in  $\mathbb{V}$ . For  $H^1(\Omega)$  conformity this is true for any  $k$ , for  $H^2(\Omega)$  one would need  $k \geq 5$  to have access to appropriate elements, as such a richer space is required. For  $k = 2$  the  $\mathbb{P}^4$  Hsieh–Clough–Tocher macro element is used with degrees of freedom given below:



Note that the reconstruction is no longer an element of  $\mathbb{V}$ .

**3.5. Definition** ( $H^1$  and  $H^2$  mesh dependant norms). We define the  $H^1$  and  $H^2$  mesh dependant norms to be

$$(3.9) \quad \|w_h\|_{dG,2}^2 := \|\nabla_h w_h\|_{L_2(\Omega)}^2 + h_e^{-1} \|[[u_h]]\|_{L_2(\mathcal{E})}^2$$

$$(3.10) \quad \|w_h\|_{dG,2}^2 := \|D_h^2 w_h\|_{L_2(\Omega)}^2 + h_e^{-1} \|[\nabla w_h]\|_{L_2(\mathcal{E})}^2 + h_e^{-3} \|[[w_h]]\|_{L_2(\mathcal{E})}^2.$$

**3.6. Lemma** (Reconstruction bounds [KP03, GHV11]). *There exist operators  $E^s : \mathbb{V} \rightarrow H^s(\Omega)$ ,  $s = 1, 2$  such that*

$$(3.11) \quad \|E^1(u_h) - u_h\|_{L_2(\Omega)}^2 \leq Ch_e \|[[u_h]]\|_{L_2(\mathcal{E})}^2$$

$$(3.12) \quad \|E^1(u_h) - u_h\|_{dG,2}^2 \leq Ch_e^{-1} \|[[u_h]]\|_{L_2(\mathcal{E})}^2$$

and

$$(3.13) \quad \|E^2(u_h) - u_h\|_{dG,2}^2 \leq C(h_e^{-1} \|[\nabla u_h]\|_{L_2(\mathcal{E})}^2 + h_e^{-3} \|[[u_h]]\|_{L_2(\mathcal{E})}^2).$$

**Proof** The proof of the bound for  $E^1(u_h)$  is given in [KP03, Thm 2.2] and for  $E^2(u_h)$  is given in [GHV11, Lem 3.1].  $\square$

**3.7. Laplace's problem.** We begin by setting  $V = W = \mathring{H}^1(\Omega)$ , which is the subspace of  $H^1(\Omega)$  with functions vanishing on  $\partial\Omega$ , and let

$$(3.14) \quad \mathcal{A}(u, v) = \int_{\Omega} \nabla u \cdot \nabla v.$$

This operator is coercive and thus certainly inf-sup stable in  $\mathring{H}^1(\Omega)$ , thus satisfies Assumption 2.1. We consider discontinuous Galerkin approximations with  $V_h = W_h = \mathbb{P}^k(\mathcal{T})$ .

The interior penalty (IP) method [Arn82]

$$(3.15) \quad \mathcal{A}_h(u_h, v_h) = \int_{\Omega} \nabla_h u_h \cdot \nabla_h v_h - \int_{\mathcal{E}} \left( [u_h] \cdot \{\nabla v_h\} + [v_h] \cdot \{\nabla u_h\} - \frac{\sigma}{h} [[u_h]] \cdot [[v_h]] \right)$$

is well known to be bounded in the  $H^1(\Omega)$  like dG-norm  $\|\cdot\|_{dG,2}$ . A conforming space  $\mathbb{W} \subset \mathring{H}^1(\Omega)$  can be found by enforcing continuity of the piecewise polynomial functions, i.e., taking  $\mathbb{W} = \mathbb{P}^p(\mathcal{T}) \cap \mathring{H}^1(\Omega)$ . Reconstruction operators  $E$  are not unique in this setting. Indeed, one example can be found in [KP03] and another the Ritz projection of  $u_h$  into  $\mathbb{P}^p(\mathcal{T}) \cap \mathring{H}^1(\Omega)$ .

**3.8. Remark** (Artificial regularity requirement). The IP method given in (3.15) is only consistent for  $u \in H^{3/2+\epsilon}(\Omega)$  since we require  $\{\nabla u\}$  to be well defined in order to invoke Strang's Lemma. This regularity is all that is required for the a posteriori analysis based on coercivity given in [KP03] for example but is insufficient for the abstract framework developped in §2, for this we would also require  $v \in H^{3/2+\epsilon}(\Omega)$  (in the evaluation of  $\mathcal{A}_h(u_h, v - v_h)$  for example) which is incompatible with the stability theory. For the bound in Theorem 2.6 to make sense it is required that the discrete bilinear form  $\mathcal{A}_h(\cdot, \cdot)$ , which is well defined over  $H^2(\mathcal{T}) \times H^2(\mathcal{T})$ , is extended to  $H^1(\mathcal{T}) \times H^1(\mathcal{T})$ , the correct domain to invoke the appropriate stability theory used for this class of PDE.

The extension is not unique. Indeed, one method of extending is to modify the definition of the bilinear form for  $u, v \in H^1(\mathcal{T})$  to

$$(3.16) \quad \mathcal{A}_h(u, v) = \int_{\Omega} \nabla_h u \cdot \nabla_h v - \int_{\mathcal{E}} \left( [u] \cdot \{P_{k-1}(\nabla v)\} + [v] \cdot \{P_{k-1}(\nabla u)\} - \frac{\sigma}{h} [u] \cdot [v] \right),$$

where  $P_{k-1}$  denotes the  $L_2$  projection into  $\mathbb{P}^{k-1}(\mathcal{T})$ . Note that this is very similar to a procedure proposed in [GIL11, §4], however extending the bilinear form in this fashion means that (3.16) coincides with (3.15) over  $\mathbb{V} \times \mathbb{V}$ , as such (3.16) is the classical IP method interpreted in such a way that a posteriori analysis can be conducted using the tools of §2.

By modifying the bilinear form in this fashion the method becomes inconsistent (in the a priori sense of (2.12)), but the inconsistency term is controllable and convergence is optimal. Another method to see this is given in [Gud10]. Here the author uses a posteriori techniques and the averaging operator from [KP03] to control the inconsistency arising from the lack of regularity up to data oscillation terms. The a posteriori bound derived under the framework of §2 is not effected by the inconsistency as we will see in the sequel.

As the IP method, the Babuška–Zlámal (BZ) method [BZ73]

$$(3.17) \quad \mathcal{A}_h(u_h, v_h) = \int_{\Omega} \nabla_h u_h \cdot \nabla_h v_h + \int_{\mathcal{E}} \frac{\sigma}{h} \llbracket u_h \rrbracket \cdot \llbracket v_h \rrbracket$$

is bounded in the dG-norm,  $\|\cdot\|_{dG,2}$ , however it is inconsistent regardless of the regularity of  $u$ , that is the second term in (2.12) is always nonzero. It is of optimal order hence, in view of Strang's Lemma, the method converges optimally. Note also that this bilinear form can be trivially extended to  $H^1(\mathcal{T}) \times H^1(\mathcal{T})$ .

**3.9. Proposition** (Approximation properties of  $L_2$  projections [DPE12, c.f. §5.6.2.2]). *Let  $P_0$  denote the  $L_2$  orthogonal projection into  $\mathbb{P}^0(\mathcal{T})$  then for any  $v \in H^1(\mathcal{T})$  we have the following local approximation bounds over elements,  $K \in \mathcal{T}$  and faces  $e \in \mathcal{E}$ :*

$$(3.18) \quad \|v - P_0 v\|_{L_2(K)} \leq Ch_K \|\nabla v\|_{L_2(K)}$$

$$(3.19) \quad \|v - P_0 v\|_{L_2(e)} \leq Ch_e^{1/2} \|\nabla v\|_{L_2(K)}.$$

**3.10. Proposition** (Approximation properties of the Scott-Zhang interpolator [EG04, c.f. §1.130]). *Let  $I_k$  denote the Scott-Zhang interpolant into  $\mathbb{V} \cap \mathring{H}^1(\Omega)$ , then the following local approximation bounds over elements,  $K \in \mathcal{T}$  and faces  $e \in \mathcal{E}$  hold:*

$$(3.20) \quad \|v - I_k v\|_{L_2(K)} \leq Ch_K \|\nabla v\|_{L_2(\widehat{K})}$$

$$(3.21) \quad \|v - I_k v\|_{L_2(e)} \leq Ch_e^{1/2} \|\nabla v\|_{L_2(\widehat{K})},$$

where  $\widehat{K}$  denotes a patch of an element  $K$ .

**3.11. Lemma** (Energy norm a posteriori control). *Let  $u_h$  be either the IP or BZ approximation of the solution of Laplace's problem, then up to a constant independent of the meshsize both approximations are a posteriori controllable by the following bound:*

$$(3.22) \quad \|u - u_h\|_{dG,2} \leq C \left( \sum_{K \in \mathcal{T}} \eta_{1,R}^2 + \sum_{e \in \mathcal{E}} \eta_{1,J}^2 \right)^{1/2} \quad \text{and}$$

with

$$(3.23) \quad \begin{aligned} \eta_{1,R}^2 &:= h_K^2 \|f + \Delta u_h\|_{L_2(K)}^2 \\ \eta_{1,J}^2 &:= h_e \|\llbracket \nabla u_h \rrbracket\|_{L_2(e)}^2 + \sigma h_e^{-1} \|\llbracket u_h \rrbracket\|_{L_2(e)}^2. \end{aligned}$$

**Proof** Apply Theorem 2.6, noting the inconsistency terms for *both* methods vanish. Then use standard a posteriori arguments [AO00, KP03, c.f.] for the residual term which we summarise here for completeness.



Firstly, for the IP method

$$\begin{aligned}
l_h(v - v_h) - \mathcal{A}_h(u_h, v - v_h) &= \int_{\Omega} f(v - v_h) - \nabla u_h \cdot (\nabla v - \nabla v_h) + \int_{\mathcal{E}} \llbracket u_h \rrbracket \cdot \{ P_{k-1}(\nabla v - \nabla v_h) \} \\
&\quad + \int_{\mathcal{E}} \llbracket v - v_h \rrbracket \{ \nabla u_h \} - \frac{\sigma}{h} \llbracket u_h \rrbracket \cdot \llbracket v - v_h \rrbracket \\
(3.24) \quad &= \int_{\Omega} (f + \Delta u_h)(v - v_h) - \int_{\mathcal{E}} \llbracket \nabla u_h \rrbracket \{ v - v_h \} \\
&\quad + \int_{\mathcal{E}} \llbracket u_h \rrbracket \cdot \{ P_{k-1}(\nabla v - \nabla v_h) \} - \frac{\sigma}{h} \llbracket u_h \rrbracket \cdot \llbracket v - v_h \rrbracket \\
&=: \mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_3 + \mathcal{R}_4.
\end{aligned}$$

Using Cauchy-Schwarz we see

$$\begin{aligned}
\mathcal{R}_1 &= \int_{\Omega} (f + \Delta u_h)(v - v_h) \\
(3.25) \quad &= \sum_{K \in \mathcal{T}} \|f + \Delta u_h\|_{L_2(K)} \|v - v_h\|_{L_2(K)} \\
&\leq C \|\nabla v\|_{L_2(\Omega)} \sum_{K \in \mathcal{T}} h_K \|f + \Delta u_h\|_{L_2(K)},
\end{aligned}$$

by taking  $v_h = P_0 v$ , the  $L_2$  orthogonal projection into  $\mathbb{P}^0(\mathcal{T})$  and using Proposition 3.9. For the second term, using the same  $v_h$

$$\begin{aligned}
\mathcal{R}_2 &= - \sum_{e \in \mathcal{E}} \int_e \llbracket \nabla u_h \rrbracket \{ v - v_h \} \\
(3.26) \quad &\leq \sum_{e \in \mathcal{E}} \|\llbracket \nabla u_h \rrbracket\|_{L_2(e)} \|\{ v - v_h \}\|_{L_2(e)} \\
&\leq C \|\nabla v\|_{L_2(\Omega)} \sum_{e \in \mathcal{E}} h_e^{1/2} \|\llbracket \nabla u_h \rrbracket\|_{L_2(e)}.
\end{aligned}$$

For the third term, in view of Cauchy Schwarz and a trace inequality and the stability of the  $L_2(\Omega)$  projection

$$\begin{aligned}
\mathcal{R}_3 &= \sum_{e \in \mathcal{E}} \int_e \llbracket u_h \rrbracket \cdot \{ P_{k-1}(\nabla v - \nabla v_h) \} \\
(3.27) \quad &\leq C \sum_{e \in \mathcal{E}} \|\llbracket u_h \rrbracket\|_{L_2(e)} \|\{ P_{k-1}(\nabla v - \nabla v_h) \}\|_{L_2(e)} \\
&\leq C \|\nabla v\|_{L_2(\Omega)} \sum_{e \in \mathcal{E}} h_e^{-1/2} \|\llbracket u_h \rrbracket\|_{L_2(e)},
\end{aligned}$$

and similarly for the final term

$$\begin{aligned}
\mathcal{R}_4 &= \sum_{e \in \mathcal{E}} \int_e \sigma h_e^{-1} \llbracket u_h \rrbracket \cdot \llbracket v - v_h \rrbracket \\
(3.28) \quad &\leq \sum_{e \in \mathcal{E}} \int_e \sigma h_e^{-1} \|\llbracket u_h \rrbracket\|_{L_2(e)} \|\llbracket v - v_h \rrbracket\|_{L_2(e)} \\
&\leq C \sigma \|\nabla v\|_{L_2(\Omega)} \sum_{e \in \mathcal{E}} h_e^{-1/2} \|\llbracket u_h \rrbracket\|_{L_2(e)}.
\end{aligned}$$

Collecting (3.30)–(3.28) we have

$$\begin{aligned}
\mathcal{E}_1 &:= \sup_{v \in \mathring{H}^1, \|\nabla v\|_{L_2(\Omega)} \leq 1} [l_h(v - v_h) - \mathcal{A}_h(u_h, v - v_h)] \\
(3.29) \quad &\leq C \left( \sum_{K \in \mathcal{T}} \eta_{1,R}^2 + \sum_{e \in \mathcal{E}} \eta_{1,J}^2 \right)^{1/2}
\end{aligned}$$

yielding the desired result. For the BZ method we note

$$\begin{aligned}
l_h(v - v_h) - \mathcal{A}_h(u_h, v - v_h) &= \int_{\Omega} f(v - v_h) - \nabla u_h \cdot (\nabla v - \nabla v_h) - \frac{\sigma}{h} \llbracket u_h \rrbracket \cdot \llbracket \nabla v - \nabla v_h \rrbracket \\
(3.30) \quad &= \int_{\Omega} (f + \Delta u_h)(v - v_h) - \int_{\mathcal{E}} \llbracket \nabla u_h \rrbracket \{ v - v_h \} \\
&\quad + \int_{\mathcal{E}} \llbracket v - v_h \rrbracket \cdot \{ \nabla u_h \} - \frac{\sigma}{h} \llbracket u_h \rrbracket \cdot \llbracket v - v_h \rrbracket \\
&=: \mathcal{R}_1 + \mathcal{R}_2 + \widetilde{\mathcal{R}}_3 + \mathcal{R}_4.
\end{aligned}$$

Notice that  $\mathcal{R}_1, \mathcal{R}_2$  and  $\mathcal{R}_4$  are the same terms that appear in the analysis of the IP method above. The third term is different due to the nature of the inconsistency of the scheme. This term vanishes as soon as  $v_h$  is chosen as a continuous function. It suffices therefore to pick  $v_h$  as the Scott–Zhang interpolator into  $\mathbb{V} \cap \mathring{H}^1(\Omega)$  and making use of Proposition 3.10. The bounds for the other terms follow analogously as above, concluding the proof.  $\square$

**3.12. Remark** (Lower bounds). It can be proven that the a posteriori bounds for both IP and BZ methods satisfy lower bounds to the error, that is, using the notation of Lemma 3.11

$$(3.31) \quad \left( \sum_{K \in \mathcal{T}} \eta_{1,R}^2 + \sum_{e \in \mathcal{E}} \eta_{1,J}^2 \right)^{1/2} \leq \|u - u_h\|_{dG,2} + h_K \|f - P_k f\|_{L_2(\Omega)}.$$

The last term in (3.31) is called a *data oscillation* term. See [KP07] for a proof of this fact for the IP method. The argument for the BZ method is identical since the a posteriori bound is the same. We will examine how other forms of inconsistency effect lower bounds in the sequel.

**3.13. Remark** (Dual norm estimates). Obtaining optimal dual norm estimates is slightly more delicate than the primal ones. Suppose the method is consistent, and in particular adjoint consistent [ABCM02], that is if  $z \in H^2(\Omega)$  solves the dual problem  $-\Delta z = g$  then

$$\mathcal{A}_h(v, z) = \int_{\Omega} v g \quad \forall v \in H^2(\mathcal{T}).$$

The IP method satisfies this condition and optimal a posteriori bounds can be obtained in the dual norm quite simply under the framework of Theorem 2.12 as the following Lemma illustrates:

**3.14. Lemma** (Adjoint consistent dual norm a posteriori control). *Let  $u_h$  be the IP approximation of the solution of Laplace’s problem, then up to a constant independent of the meshsize the following a posteriori bound holds:*

$$(3.32) \quad \|u - u_h\|_{L_2(\Omega)} \leq C \left( \sum_{K \in \mathcal{T}} \eta_{0,R}^2 + \sum_{e \in \mathcal{E}} \eta_{0,J}^2 \right)^{1/2}$$

with

$$\begin{aligned}
(3.33) \quad \eta_{0,R}^2 &:= h_K^4 \|f + \Delta u_h\|_{L_2(K)}^2 \\
\eta_{0,J}^2 &:= h_e^3 \|\llbracket \nabla u_h \rrbracket\|_{L_2(e)}^2 + \sigma h_e \|\llbracket u_h \rrbracket\|_{L_2(e)}^2.
\end{aligned}$$

**Proof** To control the nonconformity term arising in the dual a posteriori bound ( $\mathcal{E}_4^L$  in (2.24)) we note

$$\begin{aligned} \mathcal{A}_h(E^1(u_h) - u_h, z) &= \int_{\Omega} \nabla_h(E^1(u_h) - u_h) \cdot \nabla z + \int_{\mathcal{E}} \llbracket u_h \rrbracket \cdot \{\nabla z\} \\ &= \int_{\Omega} -(E^1(u_h) - u_h) \Delta z. \end{aligned} \quad (3.34)$$

Hence

$$\begin{aligned} \mathcal{E}_5 &= \sup_{g \in L, \|g\|_L \leq 1} \mathcal{A}_h(E^1(u_h) - u_h, z) \\ &= \sup_{g \in L_2(\Omega), \|g\|_{L_2(\Omega)} \leq 1} \int_{\Omega} -(E^1(u_h) - u_h) \Delta z \\ &\leq \sup_{g \in L_2(\Omega), \|g\|_{L_2(\Omega)} \leq 1} \|E^1(u_h) - u_h\|_{L_2(\Omega)} \|\Delta z\|_{L_2(\Omega)} \\ &\leq \frac{1}{\gamma^*} \|E^1(u_h) - u_h\|_{L_2(\Omega)} \\ &\leq Ch_e^{1/2} \|\llbracket u_h \rrbracket\|_{L_2(\mathcal{E})}. \end{aligned} \quad (3.35)$$

The bounds for the other terms follow from standard a posteriori arguments and can be derived by mimicking the proof of Lemma 3.11 and using regularity bounds for the solution of the dual problem.  $\square$

**3.15. Remark** (Dual bounds do not require reconstructions). To construct the dual a posteriori bound in  $L_2$  for adjoint consistent methods it is not necessary to use the reconstruction approach. The numerical approximation is smooth enough to directly apply the stability theory of the dual problem directly, at least in the case of Laplacian and biharmonic operators [GV13].

**3.16. Remark** (Dual norm estimates for inconsistent methods). For adjoint inconsistent methods, i.e., those not satisfying the condition in Remark 3.13, to obtain optimal dual norm a posteriori estimates over-penalisation is necessary, as in the a priori case [ABCM02, §5.2]. We consider the over-penalised BZ method with

$$\mathcal{A}_h(u_h, v_h) = \int_{\Omega} \nabla_h u_h \cdot \nabla_h v_h + \int_{\mathcal{E}} \sigma h_e^{-\beta} \llbracket u_h \rrbracket \cdot \llbracket v_h \rrbracket, \quad (3.36)$$

for  $\beta \geq 1$  and define the over-penalised norm as

$$\|u_h\|_{\widetilde{dG},2} := \|\nabla_h u_h\|_{L_2(\Omega)}^2 + h_e^{-\beta} \|\llbracket u_h \rrbracket\|_{L_2(\mathcal{E})}^2. \quad (3.37)$$

Optimal a posteriori bounds can be obtained for the over-penalised BZ method with  $\beta$  large enough under the framework of Theorem 2.12 as the following Lemma illustrates:

**3.17. Lemma** (Adjoint inconsistent dual norm a posteriori control). *Let  $u_h$  be the over-penalised BZ approximation of the solution of Laplace's problem, then for  $\beta \geq 3$  up to a constant independent of the meshsize the following a posteriori bound holds:*

$$\|u - u_h\|_{L_2(\Omega)} \leq C \left( \sum_{K \in \mathcal{T}} \eta_{0,R}^2 + \sum_{e \in \mathcal{E}} \eta_{0,J}^2 \right)^{1/2} \quad (3.38)$$

with

$$\begin{aligned} \eta_{0,R}^2 &:= h_K^4 \|f + \Delta u_h\|_{L_2(K)}^2 \\ \eta_{0,J}^2 &:= h_e^3 \|\llbracket \nabla u_h \rrbracket\|_{L_2(e)}^2 + \sigma h_e \|\llbracket u_h \rrbracket\|_{L_2(e)}^2. \end{aligned} \quad (3.39)$$

**Proof** The proof is analogous to Lemma 3.14 with the exception of the nonconforming term, here

$$\begin{aligned} \mathcal{A}_h(E^1(u_h) - u_h, z) &= \int_{\Omega} \nabla_h(E^1(u_h) - u_h) \cdot \nabla z \\ &= \int_{\Omega} -(E^1(u_h) - u_h) \Delta z + \int_{\mathcal{E}} \llbracket E^1(u_h) - u_h \rrbracket \cdot \{\nabla z\} \end{aligned} \quad (3.40)$$

Following the arguments of [ABCM02, §5.2] we have that

$$(3.41) \quad \begin{aligned} \int_{\mathcal{E}} \llbracket E^1(u_h) - u_h \rrbracket \cdot \llbracket \nabla z \rrbracket &\leq C h_e^{-\beta/2} \|\llbracket E^1(u_h) - u_h \rrbracket\|_{L_2(\mathcal{E})} h_e^{\beta/2} \|\llbracket \nabla z \rrbracket\|_{L_2(\mathcal{E})} \\ &\leq C \|E^1(u_h) - u_h\|_{\widetilde{dG},2} h_e^{(\beta-1)/2} \|\Delta z\|_{L_2(\Omega)} \end{aligned}$$

via a trace inequality and the definition of the over-penalised dG norm (3.37). Now through an inverse inequality we see

$$(3.42) \quad \int_{\mathcal{E}} \llbracket E^1(u_h) - u_h \rrbracket \cdot \llbracket \nabla z \rrbracket \leq C h_e^{(\beta-3)/2} \|E^1(u_h) - u_h\|_{L_2(\Omega)} \|\Delta z\|_{L_2(\Omega)},$$

and hence

$$(3.43) \quad \begin{aligned} \mathcal{E}_5 &= \sup_{g \in L, |g|_L \leq 1} \mathcal{A}_h(E^1(u_h) - u_h, z) \\ &= \sup_{g \in L_2(\Omega), \|g\|_{L_2(\Omega)} \leq 1} \left[ \int_{\Omega} -(E^1(u_h) - u_h) \Delta z + \int_{\mathcal{E}} \llbracket E^1(u_h) - u_h \rrbracket \cdot \llbracket \nabla z \rrbracket \right] \\ &\leq \sup_{g \in L_2(\Omega), \|g\|_{L_2(\Omega)} \leq 1} C \left(1 + h_e^{(\beta-3)/2}\right) \|E^1(u_h) - u_h\|_{L_2(\Omega)} \|\Delta z\|_{L_2(\Omega)} \\ &\leq \frac{C \left(1 + h_e^{(\beta-3)/2}\right)}{\gamma_*} \|E^1(u_h) - u_h\|_{L_2(\Omega)} \\ &\leq C h_e^{1/2} \|\llbracket u_h \rrbracket\|_{L_2(\mathcal{E})}, \end{aligned}$$

for  $\beta \geq 3$ , concluding the proof.  $\square$

**3.18. Quadrature approximation.** Perhaps the first topic of thought when considering notions of inconsistency in finite element analysis is that of quadrature approximations. In the case when the bilinear form contains a positive definite, non constant diffusion tensor inconsistencies can arise from the inexact integration [Cia78, §4.1]. Take  $\mathbf{A} \in L_{\infty}(\Omega)^{d \times d}$  and

$$(3.44) \quad \mathcal{A}(u, v) = \int_{\Omega} (\mathbf{A} \nabla u) \cdot \nabla v,$$

for example, then both linear and bilinear forms are practically approximated by using quadratures.

**3.19. Definition (Quadrature).** We introduce

$$(3.45) \quad Q_K^s(f) := \sum_{i_q=1}^{n_q} w_{i_q} f(x_{i_q})$$

to be a quadrature where  $\{(x_{i_q}, w_{i_q})\}_{i_q=1}^{n_q}$  denote the positions and weights of the quadrature points over an element  $K$ . This is said to be an order  $s$  quadrature if

$$(3.46) \quad \int_K f = Q_K^s(f) \quad \forall f \in \mathbb{P}^s(K)$$

and

$$(3.47) \quad \left| \int_K f(x) dx - Q_K^s(f) \right| \leq C h^{s+1} |f|_{H^s(K)} \quad \forall f \in H^s(K).$$

Let us, for simplicity, temporarily consider the  $C^0$  conforming method under quadrature approximation applied to (3.44) which reads: Find  $u_h \in \mathbb{V} \cap \mathring{H}^1(\Omega)$  such that

$$(3.48) \quad \sum_{K \in \mathcal{T}} Q_K^s((\mathbf{A} \nabla u_h) \cdot \nabla v_h) = \sum_{K \in \mathcal{T}} Q_K^s(f v_h) \quad \forall v_h \in \mathbb{V} \cap \mathring{H}^1(\Omega).$$

**3.20. Remark** (Standard quadrature degree choice). A relatively standard choice of quadrature degree is  $s = 2k - 2$ . This is because if  $\mathbf{A}$  is constant then the quadrature allows for exact evaluation of the bilinear form. The linear form however is not, in general, integrated exactly. Indeed, the linear form can be interpreted as

$$(3.49) \quad l_h(v_h) = \sum_{K \in \mathcal{T}} Q_K^{2k-2}(fv_h) = \sum_{K \in \mathcal{T}} \int_K P_{k-2} f v_h,$$

where  $P_{k-2}$  denotes the  $L_2$  projection into  $\mathbb{P}^{\max(k-2,0)}(\mathcal{T})$ .

**3.21. Lemma** (A posteriori bound for quadrature approximations). *Let  $u \in \mathring{H}^1(\Omega)$  be a weak solution to the variational problem*

$$(3.50) \quad \mathcal{A}(u, v) = l(v) \quad \forall v \in \mathring{H}^1(\Omega),$$

with  $\mathcal{A}(u, v)$  given by (3.44) and  $u_h$  solve (3.48) with quadrature degree  $2k - 2$ , then

$$(3.51) \quad \|u - u_h\|_{H^1(\Omega)} \leq C \left( \sum_{K \in \mathcal{T}} \eta_R^2 + \eta_I^2 + \sum_{e \in \mathcal{E}} \eta_J^2 \right)^{1/2},$$

where

$$(3.52) \quad \begin{aligned} \eta_R^2 &:= h_K^2 \|P_{k-2} f + \operatorname{div}(P_{k-1}(\mathbf{A} \nabla u_h))\|_{L_2(K)}^2 \\ \eta_I^2 &:= h_K^2 \|f - P_{k-2} f\|_{L_2(K)}^2 + \|(P_{k-1} - \operatorname{Id})(\mathbf{A} \nabla u_h)\|_{L_2(K)}^2 \\ \eta_J^2 &:= h_e \|[P_{k-1}(\mathbf{A} \nabla u_h)]\|_{L_2(e)}^2. \end{aligned}$$

**3.22. Remark** (Evaluation of the estimator and data oscillation). The indicator components  $\eta_R, \eta_J$  have the standard form of an a posteriori estimator with the exception that they involve *projections* of the problem data. This allows for these terms to be evaluated *exactly* using the same quadrature as used in the scheme, that is, there is no additional approximation involved in the computation of these terms.

The estimator  $\eta_I$ , representing the inconsistency, is a data oscillation term. In general, the integration required in computation of the estimator *cannot* be performed exactly using the quadrature degree in the scheme. If some smoothness is assumed on the problem data these terms can be considered of higher order in comparison to the residual, however in practical computations resolution of the problem data is extremely important especially at coarse mesh scales, see [BDN13]. In the design of convergent adaptive algorithms the inclusion of these terms in the marking strategy is of paramount importance see [MNS02, MN05] for further arguments.

**Proof of Lemma 3.21** Since the method we consider is conforming Theorem 2.6 simplifies to give

$$(3.53) \quad \begin{aligned} \|\nabla u - \nabla u_h\|_{L_2(\Omega)} &\leq \frac{1}{\gamma} \left( \sup_{v \in \mathring{H}^1, \|v\|_{\mathring{H}^1} \leq 1} [l_h(v - v_h) - \mathcal{A}_h(u_h, v - v_h)] + \sup_{v \in \mathring{H}^1, \|v\|_{\mathring{H}^1} \leq 1} [l(v) - l_h(v)] \right. \\ &\quad \left. + \sup_{v \in \mathring{H}^1, \|v\|_{\mathring{H}^1} \leq 1} [\mathcal{A}_h(u_h, v) - \mathcal{A}(u_h, v)] \right) \\ &=: \frac{1}{\gamma} (\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3). \end{aligned}$$

The first term is a standard residual term and can be controlled using standard a posteriori techniques. Indeed, using (3.49) we interpret

$$(3.54) \quad l_h(v - v_h) = \sum_{K \in \mathcal{T}} \int_K P_{k-2} f (v - v_h).$$

Similarly,

$$(3.55) \quad \mathcal{A}_h(u_h, v - v_h) = \sum_{K \in \mathcal{T}} \int_K P_{k-1}(\mathbf{A} \nabla u_h) \nabla(v - v_h).$$

Hence

$$\begin{aligned}
l_h(v - v_h) - \mathcal{A}_h(u_h, v - v_h) &= \sum_{K \in \mathcal{T}} \left[ \int_K (P_{k-2}f + \operatorname{div}(P_{k-1}(\mathbf{A}\nabla u_h)))(v - v_h) \right. \\
&\quad \left. - \int_{\partial K} P_{k-1}(\mathbf{A}\nabla u_h) \cdot \mathbf{n}(v - v_h) \right] \\
(3.56) \qquad &= \sum_{K \in \mathcal{T}} \left[ \int_K (P_{k-2}f + \operatorname{div}(P_{k-1}(\mathbf{A}\nabla u_h)))(v - v_h) \right] \\
&\quad - \int_{\mathcal{E}} \llbracket P_{k-1}(\mathbf{A}\nabla u_h) \rrbracket (v - v_h).
\end{aligned}$$

Taking  $v_h$  to be the Scott-Zhang interpolant of  $v$  and using the approximability properties of Proposition 3.10 we have

$$(3.57) \qquad \mathcal{E}_1 \leq C \sum_{K \in \mathcal{T}} \left( h_K \|P_{k-2}f + \operatorname{div}(P_{k-1}(\mathbf{A}\nabla u_h))\|_{L_2(K)} + \sum_{e \in K} h_e^{1/2} \|\llbracket P_{k-1}(\mathbf{A}\nabla u_h) \rrbracket\|_{L_2(e)} \right).$$

As for the other term, note that

$$\begin{aligned}
\mathcal{E}_2 &= \sup_{v \in \mathring{H}^1, \|v\|_{\mathring{H}^1} \leq 1} [l(v) - l_h(v)] \\
&= \sup_{v \in \mathring{H}^1, \|v\|_{\mathring{H}^1} \leq 1} \int_{\Omega} (f - P_{k-2}f) v \\
(3.58) \qquad &= \sup_{v \in \mathring{H}^1, \|v\|_{\mathring{H}^1} \leq 1} \int_{\Omega} (f - P_{k-2}f)(v - P_{k-2}v) \\
&\leq \sup_{v \in \mathring{H}^1, \|v\|_{\mathring{H}^1} \leq 1} \sum_{K \in \mathcal{T}} \|f - P_{k-2}f\|_{L_2(K)} \|v - P_{k-2}v\|_{L_2(K)} \\
&\leq C \sum_{K \in \mathcal{T}} h_K \|f - P_{k-2}f\|_{L_2(K)},
\end{aligned}$$

using the approximability of the  $L_2$  projection operator from Proposition 3.9.

The final bound follows from definition since

$$\begin{aligned}
\mathcal{E}_3 &= \sup_{v \in \mathring{H}^1, \|v\|_{\mathring{H}^1} \leq 1} [\mathcal{A}_h(u_h, v) - \mathcal{A}(u_h, v)] \\
(3.59) \qquad &= \sup_{v \in \mathring{H}^1, \|v\|_{\mathring{H}^1} \leq 1} \sum_{K \in \mathcal{T}} \int_K (P_{k-1} - \operatorname{Id})(\mathbf{A}\nabla u_h) \nabla v. \\
&\leq \sum_{K \in \mathcal{T}} \|(P_{k-1} - \operatorname{Id})(\mathbf{A}\nabla u_h)\|_{L_2(K)},
\end{aligned}$$

by Cauchy-Schwarz. Collecting (3.57), (3.58) and (3.59) yields the desired result.  $\square$

**3.23. Proposition** (A posteriori lower bound under quadrature approximation). *Let  $u$ ,  $u_h$  and  $\eta_R, \eta_J$  be as in Lemma 3.21 then*

$$(3.60) \qquad \eta_R + \sum_{e \in K} \eta_J \leq C \left( \|\nabla u - \nabla u_h\|_{L_2(\widehat{K})} + \eta_I \right),$$

where  $\widehat{K}$  denotes the set of all elements sharing a common edge with  $K$ .

**Proof** The proof of this fact is relatively standard, we will include the first part, that of the element residual, for completeness and to compare with a result in a later section. Throughout this proof we will use the convention that  $a \lesssim b$  means  $a \leq Cb$  where  $C$  is a generic constant that may depend on the problem

data, but is independent of the meshsize  $u$  and  $u_h$ . We begin by defining the bubble functions. Let  $b_K$  be the interior bubble function defined on the reference triangle  $K_{\text{ref}}$  with barycentric coordinates  $\{\lambda_i\}_{i=0}^d$  through  $b_{K_{\text{ref}}} = (d+1)^{d+1} \prod_{i=0}^d \lambda_i$ . Due to [Ver96] these functions satisfy

$$(3.61) \quad \|b_K v\|_{L_2(K)} \leq \|v\|_{L_2(K)} \lesssim \|b_K v\|_{L_2(K)} \quad \text{for } v \in \mathbb{P}^k(K).$$

In view of this we have that

$$(3.62) \quad \|P_{k-2}f + \text{div}(P_{k-1}(\mathbf{A}\nabla u_h))\|_{L_2(K)}^2 \lesssim \|b_K^{1/2}(P_{k-2}f + \text{div}(P_{k-1}(\mathbf{A}\nabla u_h)))\|_{L_2(K)}^2,$$

so

$$(3.63) \quad \begin{aligned} \|P_{k-2}f + \text{div}(P_{k-1}(\mathbf{A}\nabla u_h))\|_{L_2(K)}^2 &\lesssim \int_K (P_{k-2}f - f) b_K (P_{k-2}f + \text{div}(P_{k-1}(\mathbf{A}\nabla u_h))) \\ &\quad + \int_K (f + \text{div}(\mathbf{A}\nabla u_h)) b_K (P_{k-2}f + \text{div}(P_{k-1}(\mathbf{A}\nabla u_h))) \\ &\quad + \int_K (\text{div}(P_{k-1}(\mathbf{A}\nabla u_h)) - \text{div}(\mathbf{A}\nabla u_h)) b_K (P_{k-2}f + \text{div}(P_{k-1}(\mathbf{A}\nabla u_h))) \\ &\lesssim \int_K (P_{k-2}f - f) b_K (P_{k-2}f + \text{div}(P_{k-1}(\mathbf{A}\nabla u_h))) \\ &\quad + \int_K (\mathbf{A}\nabla u_h - \mathbf{A}\nabla u) \cdot \nabla (b_K (P_{k-2}f + \text{div}(P_{k-1}(\mathbf{A}\nabla u_h)))) \\ &\quad + \int_K (P_{k-1}(\mathbf{A}\nabla u_h) - \mathbf{A}\nabla u_h) \cdot \nabla (b_K (P_{k-2}f + \text{div}(P_{k-1}(\mathbf{A}\nabla u_h)))) \end{aligned}$$

as  $b_K$  vanishes on  $\partial K$ . Now using Cauchy-Schwarz, further properties of  $b_K$  and inverse inequalities

$$(3.64) \quad \begin{aligned} \|P_{k-2}f + \text{div}(P_{k-1}(\mathbf{A}\nabla u_h))\|_{L_2(K)} &\lesssim \|P_{k-2}f - f\|_{L_2(K)} + h_K^{-1} \|\mathbf{A}\nabla u - \mathbf{A}\nabla u_h\|_{L_2(K)} \\ &\quad + h_K^{-1} \|\mathbf{A}\nabla u_h - P_{k-1}(\mathbf{A}\nabla u_h)\|_{L_2(K)} \\ &\lesssim \|P_{k-2}f - f\|_{L_2(K)} + h_K^{-1} \|\nabla u - \nabla u_h\|_{L_2(K)} \\ &\quad + h_K^{-1} \|\mathbf{A}\nabla u_h - P_{k-1}(\mathbf{A}\nabla u_h)\|_{L_2(K)}. \end{aligned}$$

To see that the jump residual behaves in a similar way one may follow the arguments in [EG04, Proof of Thm 10.10] with a slight modification to take into account the data approximation.  $\square$

The extension to a nonconforming method is almost immediate using the reconstruction operator from [KP03] and the arguments previously. For instance, consider the problem to seek  $u_h \in \mathbb{V}$  such that

$$(3.65) \quad \mathcal{A}_h(u_h, v_h) = l_h(v_h) \quad \forall v_h \in \mathbb{V},$$

where

$$(3.66) \quad \begin{aligned} \mathcal{A}_h(u_h, v_h) &:= \sum_{K \in \mathcal{T}} Q_K^{2k-2}((\mathbf{A}\nabla u_h) \cdot \nabla v_h) - \sum_{e \in \mathcal{E}} \left[ Q_e^{2k-1}(\llbracket u_h \rrbracket \cdot \llbracket \mathbf{A}\nabla v_h \rrbracket) \right. \\ &\quad \left. + Q_e^{2k-1}(\llbracket v_h \rrbracket \cdot \llbracket \mathbf{A}\nabla u_h \rrbracket) - \frac{\sigma}{h} Q_e^{2k}(\llbracket v_h \rrbracket \cdot \llbracket u_h \rrbracket) \right] \end{aligned}$$

and  $l_h(v_h)$  is given by (3.49). This is an example of an IP method after quadrature applied to (3.44). Notice the difference in quadrature degree chosen to integrate over edges. In addition note the similarity in structure to that of (3.16).

**3.24. Corollary** (A posteriori bound for nonconforming schemes including quadrature approximation). *Let  $u$  be a weak solution of (3.44) and  $u_h$  solve (3.65), then*

$$(3.67) \quad \|u - u_h\|_{dG,2} \leq \left( \sum_{K \in \mathcal{T}} \eta_R^2 + \widetilde{\eta}_I^2 + \sum_{e \in \mathcal{E}} \eta_J^2 \right)^{1/2}$$

where

$$\begin{aligned}
\eta_R^2 &:= h_K^2 \|P_{k-2}f + \operatorname{div}(P_{k-1}(\mathbf{A}\nabla u_h))\|_{L_2(K)}^2 \\
(3.68) \quad \widetilde{\eta}_I^2 &:= h_K^2 \|f - P_{k-2}f\|_{L_2(K)}^2 + \|(P_{k-1} - \operatorname{Id})(\mathbf{A}\nabla E^1(u_h))\|_{L_2(K)}^2 \\
\eta_J^2 &:= h_e \|P_{k-1}(\mathbf{A}\nabla u_h)\|_{L_2(e)}^2 + \sigma h_e^{-1} \|\llbracket u_h \rrbracket\|_{L_2(e)}^2.
\end{aligned}$$

Also for  $\sigma$  large enough

$$(3.69) \quad \eta_R + \sum_{e \in K} \eta_J \leq C(\|\nabla u - \nabla u_h\|_{L_2(K)} + \eta_I)$$

with

$$(3.70) \quad \eta_I^2 := h_K^2 \|f - P_{k-2}f\|_{L_2(K)}^2 + \|(P_{k-1} - \operatorname{Id})(\mathbf{A}\nabla u_h)\|_{L_2(K)}^2$$

**Proof** The main difference between control of the conforming approximation and the nonconforming is that of the penalty terms. To see these control the error from below we modify the argument given in [DPE12, Lem 5.30]. The main idea is that for variational second order problems there is a conformal reconstruction in the dG space. This allows us to make use of the numerical scheme in the following fashion. We interpret (3.66) as

$$\begin{aligned}
(3.71) \quad \mathcal{A}_h(u_h, v_h) &= \int_{\Omega} P_{k-1}(\mathbf{A}\nabla_h u_h) \cdot \nabla_h v_h \\
&\quad - \int_{\mathcal{E}} \left[ \llbracket u_h \rrbracket \{ P_{k-1}(\mathbf{A}\nabla v_h) \} + \llbracket v_h \rrbracket \{ P_{k-1}(\mathbf{A}\nabla u_h) \} - \sigma h_e^{-1} \llbracket u_h \rrbracket \cdot \llbracket v_h \rrbracket \right],
\end{aligned}$$

which, using the fact that  $u_h$  solves (3.65), allows us to choose  $v_h = u_h - E^1(u_h)$ . From this we see, since  $\llbracket E^1(u_h) \rrbracket = 0$ , that after an integration by parts

$$\begin{aligned}
(3.72) \quad \sigma h_e^{-1} \|\llbracket u_h \rrbracket\|_{L_2(\mathcal{E})}^2 &= \int_{\Omega} (P_{k-2}f + \operatorname{div}(P_{k-1}(\mathbf{A}\nabla u_h)))(u_h - E^1(u_h)) \\
&\quad - \int_{\mathcal{E}} \left[ \llbracket P_{k-1}(\mathbf{A}\nabla u_h) \rrbracket \cdot \{ u_h - E^1(u_h) \} - \llbracket u_h \rrbracket \cdot \{ P_{k-1}(\mathbf{A}\nabla(u_h - E^1(u_h))) \} \right] \\
&\leq \|P_{k-2}f + \operatorname{div}(P_{k-1}(\mathbf{A}\nabla u_h))\|_{L_2(\Omega)} \|u_h - E^1(u_h)\|_{L_2(\Omega)} \\
&\quad + \|\llbracket P_{k-1}(\mathbf{A}\nabla u_h) \rrbracket\|_{L_2(\mathcal{E})} \|\{ u_h - E^1(u_h) \}\|_{L_2(\mathcal{E})} \\
&\quad + \|\llbracket u_h \rrbracket\|_{L_2(\mathcal{E})} \|\{ P_{k-1}(\mathbf{A}\nabla(u_h - E^1(u_h))) \}\|_{L_2(\mathcal{E})} \\
&=: \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3.
\end{aligned}$$

The first term, owing to the properties of the reconstruction  $E^1(u_h)$  is controlled by

$$\begin{aligned}
(3.73) \quad \mathcal{J}_1 &= \|P_{k-2}f + \operatorname{div}(P_{k-1}(\mathbf{A}\nabla u_h))\|_{L_2(\Omega)} \|u_h - E^1(u_h)\|_{L_2(\Omega)} \\
&\leq Ch_e^{1/2} \|P_{k-2}f + \operatorname{div}(P_{k-1}(\mathbf{A}\nabla u_h))\|_{L_2(\Omega)} \|\llbracket u_h \rrbracket\|_{L_2(\mathcal{E})}.
\end{aligned}$$

The second term, via a trace inequality

$$\begin{aligned}
(3.74) \quad \mathcal{J}_2 &= \|\llbracket P_{k-1}(\mathbf{A}\nabla u_h) \rrbracket\|_{L_2(\mathcal{E})} \|\{ u_h - E^1(u_h) \}\|_{L_2(\mathcal{E})} \\
&\leq Ch_e^{-1/2} \|\llbracket P_{k-1}(\mathbf{A}\nabla u_h) \rrbracket\|_{L_2(\mathcal{E})} \|u_h - E^1(u_h)\|_{L_2(\Omega)} \\
&\leq C \|\llbracket P_{k-1}(\mathbf{A}\nabla u_h) \rrbracket\|_{L_2(\mathcal{E})} \|\llbracket u_h \rrbracket\|_{L_2(\mathcal{E})}.
\end{aligned}$$

Finally the third term, again by a trace inequality and the stability of the  $L_2$  projection

$$\begin{aligned}
(3.75) \quad \mathcal{J}_3 &= \|\llbracket u_h \rrbracket\|_{L_2(\mathcal{E})} \|\{ P_{k-1}(\mathbf{A}\nabla(u_h - E^1(u_h))) \}\|_{L_2(\mathcal{E})} \\
&\leq Ch_e^{-1/2} \|\llbracket u_h \rrbracket\|_{L_2(\mathcal{E})} \|P_{k-1}(\mathbf{A}\nabla(u_h - E^1(u_h)))\|_{L_2(\Omega)} \\
&\leq Ch_e^{-1/2} \|\llbracket u_h \rrbracket\|_{L_2(\mathcal{E})} \|\mathbf{A}\nabla(u_h - E^1(u_h))\|_{L_2(\Omega)} \\
&\leq \widetilde{C} h_e^{-1} \|\llbracket u_h \rrbracket\|_{L_2(\mathcal{E})}^2.
\end{aligned}$$



Substituting (3.73), (3.74) and (3.75) into (3.72) yields

$$(3.76) \quad (\sigma - \widetilde{C}) h_e^{-1} \|\llbracket u_h \rrbracket\|_{L_2(\mathcal{E})}^2 \leq C \left( \|P_{k-2}f + \operatorname{div}(P_{k-1}(\mathbf{A}\nabla u_h))\|_{L_2(\Omega)} + \|\llbracket P_{k-1}(\mathbf{A}\nabla u_h) \rrbracket\|_{L_2(\mathcal{E})} \right).$$

Hence for  $\sigma > \widetilde{C}$  the penalty term is controlled by the other residual terms which in turn, using the arguments from Proposition 3.23, are controlled by the error up to the data oscillation terms given by  $\eta_I$ , concluding the proof.  $\square$

**3.25. Remark** (Inconsistencies require computation of reconstructions). In the statement of the bound for the nonconforming approximation of the Laplacian in Lemma 3.11 there is no requirement to compute the reconstruction operator,  $E^1$ . Indeed, if there is no inconsistency in the approximation of the nonconstant diffusion problem (3.44) then it is still not necessary to actually compute the reconstruction operator. The nature of the inconsistency term  $\widehat{\eta}_I$  appearing in Corollary 3.24 requires computation of the reconstruction of  $u_h$ . We will see in a later section that inconsistencies do not *always* require computation of this operator.

#### 4. UNBALANCED VARIATIONAL AND NONVARIATIONAL PROBLEMS

In this section we will take a different route. For simplicity initially we shall ignore any possible effect of quadrature and focus on the effects of inconsistencies in unbalanced problems. We will consider the case  $W = H^2(\Omega) \cap \mathring{H}^1(\Omega)$  and  $V = L_2(\Omega)$ . Let

$$(4.1) \quad \mathcal{A}(u, v) = \int_{\Omega} \mathbf{A} : D^2 u,$$

where  $\mathbf{A} \in \mathbb{R}^{d \times d}$  is positive definite diffusion tensor,  $D^2 u$  denotes the Hessian matrix of  $u$  and  $\mathbf{X} : \mathbf{Y} = \operatorname{trace}(\mathbf{X}^T \mathbf{Y})$  is the Frobenius product between matrices. The operator (4.1) is in nondivergence form. Galerkin schemes falling into the framework presented here are given in [LP11, DP13]. Both are nonconforming, inconsistent approximation schemes.

**4.1. Assumption** (Strong solutions exist). Under either smoothness assumptions on  $\mathbf{A}$  [GT83], Campanato [Cam94] or Cordes [Cor61] conditions on  $\mathbf{A}$  the problem

$$(4.2) \quad \mathcal{A}(u, v) = \int_{\Omega} f v$$

has a strong solution, that is, there exists a constant  $C > 0$  such that

$$(4.3) \quad \|u\|_{H^2(\Omega)} \leq C \|f\|_{L_2(\Omega)}.$$

This is equivalent to Assumption 2.1. Note that if  $\mathbf{A} = -\mathbf{I}$  then this is a natural framework to study the Laplacian if it has strong solutions.

**4.2. Consistent nonvariational methods.** Henceforth we will take  $k \geq 2$ . For  $\mathbf{A} \in L_{\infty}(\Omega)^{d \times d}$  satisfying Assumption 4.1 we define

$$(4.4) \quad \mathcal{A}_h(u_h, v_h) = \int_{\Omega} \mathbf{A} : D_h^2 u_h v_h - \int_{\mathcal{E}} (\llbracket \nabla u_h \rrbracket_{\otimes} : \llbracket \mathbf{A} v_h \rrbracket - \sigma h_e \llbracket \nabla u_h \rrbracket \llbracket \nabla v_h \rrbracket - \sigma h_e^{-1} \llbracket u_h \rrbracket \cdot \llbracket v_h \rrbracket),$$

and seek  $u_h \in \mathbb{V}$  such that

$$(4.5) \quad \mathcal{A}_h(u_h, v_h) = \int_{\Omega} f v_h \quad \forall v_h \in \mathbb{V}.$$

Note for  $u \in H^2(\Omega)$  the method is consistent, in that

$$(4.6) \quad \mathcal{A}_h(u, v) = \mathcal{A}(u, v) \quad \forall v \in H^2(\mathcal{T}).$$

**4.3. Remark** (Extension of the bilinear form). The discrete bilinear form (4.4) only makes sense over  $H^2(\mathcal{T}) \times H^2(\mathcal{T})$ . To make use of the a posteriori framework in §2 we require an extension to ensure the appropriate stability arguments can be applied. That is we require the bilinear form to be extended to  $H^2(\mathcal{T}) \times L_2(\Omega)$ . To do this for  $(u, v) \in H^2(\mathcal{T}) \times L_2(\Omega)$  we define

$$(4.7) \quad \mathcal{A}_h(u, v) = \int_{\Omega} \mathbf{A} : D_h^2 u v - \int_{\mathcal{E}} (\llbracket \nabla u \rrbracket_{\otimes} : \llbracket \mathbf{A} P_k v \rrbracket - \sigma h_e \llbracket \nabla u \rrbracket \llbracket \nabla(P_k v) \rrbracket - \sigma h_e^{-1} \llbracket u \rrbracket \cdot \llbracket P_k v \rrbracket).$$

Notice that the modified bilinear form (4.7) coincides with (4.4) over  $\mathbb{V} \times \mathbb{V}$  and that it satisfies

$$(4.8) \quad \mathcal{A}_h(u, v) \leq C \|u\|_{dG,2} \|v\|_{L_2(\Omega)} \quad \text{for } (u, v) \in (H^2(\mathcal{T}) \times L_2(\Omega)).$$

**4.4. Theorem** (Primal consistent a posteriori upper bound). *Let  $u \in H^2(\Omega)$  solve (4.2) and  $u_h \in \mathbb{V}$  be the finite element approximation given by (4.5), then*

$$(4.9) \quad \|u - u_h\|_{dG,2} \leq C \left( \sum_{K \in \mathcal{T}} \eta_R^2 + \sum_{e \in \mathcal{E}} \eta_J^2 \right)^{1/2},$$

where

$$(4.10) \quad \eta_R^2 := \|f - \mathbf{A}:\mathbf{D}^2 u_h\|_{L_2(K)}^2$$

$$(4.11) \quad \eta_J^2 := h_e^{-1} \|\llbracket \nabla u_h \rrbracket\|_{L_2(e)}^2 + h_e^{-3} \|\llbracket u_h \rrbracket\|_{L_2(e)}^2.$$

**Proof** Making use of the modified bilinear form given in Remark 4.3 we may apply the framework of Theorem 2.6 directly here and we have

$$(4.12) \quad \begin{aligned} \|e\|_{H^2(\Omega)} &\leq \frac{1}{\gamma} \left( \sup_{v \in L_2(\Omega), \|v\|_{L_2(\Omega)} \leq 1} [l_h(v - v_h) - \mathcal{A}_h(u_h, v - v_h)] + \sup_{v \in L_2(\Omega), \|v\|_{L_2(\Omega)} \leq 1} [l(v) - l_h(v)] \right. \\ &\quad \left. + \sup_{v \in L_2(\Omega), \|v\|_{L_2(\Omega)} \leq 1} [\mathcal{A}_h(E^2(u_h), v) - \mathcal{A}(E^2(u_h), v)] \right) + \left(1 + \frac{C_B}{\gamma}\right) \|e^N\|_{H^2(\Omega)} \\ &=: \frac{1}{\gamma} (\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3 + (\gamma + C_B) \mathcal{E}_4). \end{aligned}$$

Note that  $\mathcal{E}_2 = 0$  and since  $E^2(u_h) \in H^2(\Omega)$  all jump terms from (4.7) vanish and we see  $\mathcal{E}_3 = 0$ . To control  $\mathcal{E}_1$  we see in view of Cauchy-Schwarz

$$(4.13) \quad \begin{aligned} \mathcal{E}_1 &= \sup_{v \in L_2(\Omega), \|v\|_{L_2(\Omega)} \leq 1} [l_h(v - v_h) - \mathcal{A}_h(u_h, v - v_h)] \\ &= \sup_{v \in L_2(\Omega), \|v\|_{L_2(\Omega)} \leq 1} \left[ \int_{\Omega} (f - \mathbf{A}:\mathbf{D}_h^2 u_h)(v - v_h) + \int_{\mathcal{E}} \llbracket \nabla u_h \rrbracket_{\otimes} : \llbracket \mathbf{A} P_k(v - v_h) \rrbracket \right. \\ &\quad \left. - \sigma h_e \int_{\mathcal{E}} \llbracket \nabla u_h \rrbracket \llbracket \nabla (P_k(v - v_h)) \rrbracket - \sigma h_e^{-1} \int_{\mathcal{E}} \llbracket u_h \rrbracket \cdot \llbracket P_k(v - v_h) \rrbracket \right] \\ &\leq \sup_{v \in L_2(\Omega), \|v\|_{L_2(\Omega)} \leq 1} \left[ \sum_{K \in \mathcal{T}} \|f - \mathbf{A}:\mathbf{D}_h^2 u_h\|_{L_2(K)} \|v - v_h\|_{L_2(K)} \right. \\ &\quad \left. + \sum_{e \in \mathcal{E}} \left( \|\llbracket \nabla u_h \rrbracket\|_{L_2(e)} \left( \|\llbracket \mathbf{A} P_k(v - v_h) \rrbracket\|_{L_2(e)} + \sigma h_e \|\llbracket \nabla (P_k(v - v_h)) \rrbracket\|_{L_2(e)} \right) \right. \right. \\ &\quad \left. \left. + \sigma h_e^{-1} \|\llbracket u_h \rrbracket\|_{L_2(e)} \|\llbracket P_k(v - v_h) \rrbracket\|_{L_2(e)} \right) \right]. \end{aligned}$$

Now choosing  $v_h = 0$ , using trace and inverse inequalities and the stability of  $P_k$  in  $L_2$  we have

$$\begin{aligned}
(4.14) \quad \mathcal{E}_1 &\leq \sup_{v \in L_2(\Omega), \|v\|_{L_2(\Omega)} \leq 1} \sum_{K \in \mathcal{T}} \left( \|f - \mathbf{A} : \mathbf{D}_h^2 u_h\|_{L_2(K)} \|v\|_{L_2(K)} \right. \\
&\quad + C \sum_{e \in K} \left( \|\llbracket \nabla u_h \rrbracket\|_{L_2(e)} (h_e^{-1/2} \|P_k(v)\|_{L_2(K)} + \sigma h_e^{1/2} \|\nabla P_k v\|_{L_2(K)}) \right. \\
&\quad \left. \left. + \|\llbracket u_h \rrbracket\|_{L_2(e)} h_e^{-3/2} \|P_k v\|_{L_2(K)} \right) \right) \\
&\leq \sup_{v \in L_2(\Omega), \|v\|_{L_2(\Omega)} \leq 1} \|v\|_{L_2(\Omega)} \sum_{K \in \mathcal{T}} \left( \|f - \mathbf{A} : \mathbf{D}_h^2 u_h\|_{L_2(K)} \right. \\
&\quad \left. + C \sum_{e \in K} \left( C h_e^{-1/2} \|\llbracket \nabla u_h \rrbracket\|_{L_2(e)} + h_e^{-3/2} \|\llbracket u_h \rrbracket\|_{L_2(e)} \right) \right) \\
&\leq \sum_{K \in \mathcal{T}} \left( \|f - \mathbf{A} : \mathbf{D}_h^2 u_h\|_{L_2(K)} + C \sum_{e \in K} \left( C h_e^{-1/2} \|\llbracket \nabla u_h \rrbracket\|_{L_2(e)} + h_e^{-3/2} \|\llbracket u_h \rrbracket\|_{L_2(e)} \right) \right).
\end{aligned}$$

To conclude we use the reconstruction bounds given in Lemma 3.6 to control  $\mathcal{E}_4$  as required.  $\square$

**4.5. Proposition** (Lower bounds). *Under the conditions of Theorem 4.4 the following global a posteriori lower bound holds:*

$$(4.15) \quad \sum_{K \in \mathcal{T}} \eta_R + \sum_{e \in \mathcal{E}} \eta_J \leq \|D^2 u - D_h^2 u_h\|_{L_2(\Omega)},$$

where

$$(4.16) \quad \eta_R^2 := \|f - \mathbf{A} : \mathbf{D}^2 u_h\|_{L_2(K)}^2$$

$$(4.17) \quad \eta_J^2 := h_e^{-1} \|\llbracket \nabla u_h \rrbracket\|_{L_2(e)}^2 + h_e^{-3} \|\llbracket u_h \rrbracket\|_{L_2(e)}^2.$$

**Proof** The bound on the interior residual is trivial since the error and residual are being measured in the same norm there is no need to use bubble functions to invoke inverse inequalities. Indeed,

$$(4.18) \quad \|f - \mathbf{A} : \mathbf{D}_h^2 u_h\|_{L_2(\Omega)} \leq \|A\|_{L_\infty(\Omega)} \|D^2 u - D_h^2 u_h\|_{L_2(\Omega)}.$$

For the jump terms we may use a similar argument to that given in the proof of Corollary 3.24. We choose  $v_h = u_h - E^1(u_h)$  in (4.4) and then by definition of the scheme

$$\begin{aligned}
(4.19) \quad \sigma h_e^{-1} \|\llbracket u_h \rrbracket\|_{L_2(e)}^2 &\leq \sigma h_e^{-1} \|\llbracket \nabla u_h \rrbracket\|_{L_2(\mathcal{E})} \|\llbracket \nabla u_h - \nabla E^1(u_h) \rrbracket\|_{L_2(\mathcal{E})} + \|\llbracket \nabla u_h \rrbracket\|_{L_2(\mathcal{E})} \|\llbracket u_h - E^1(u_h) \rrbracket\|_{L_2(\mathcal{E})} \\
&\quad + \|f - \mathbf{A} : \mathbf{D}_h^2 u_h\|_{L_2(\Omega)} \|u_h - E^1(u_h)\|_{L_2(\Omega)}.
\end{aligned}$$

Now using the properties of  $E^1$  from Lemma 3.6 we have

$$(4.20) \quad \sigma h_e^{-1} \|\llbracket u_h \rrbracket\|_{L_2(\mathcal{E})}^2 \leq C \sigma \|\llbracket \nabla u_h \rrbracket\|_{L_2(\mathcal{E})} \|\llbracket u_h \rrbracket\|_{L_2(\mathcal{E})} + h_e^{1/2} \|f - \mathbf{A} : \mathbf{D}_h^2 u_h\|_{L_2(K)} \|\llbracket u_h \rrbracket\|_{L_2(\mathcal{E})}.$$

Now using the fact that

$$(4.21) \quad h_e^{-1} \|\llbracket \nabla u_h \rrbracket\|_{L_2(\mathcal{E})}^2 = h_e^{-1} \|\llbracket \nabla u - \nabla u_h \rrbracket\|_{L_2(\mathcal{E})}^2 \leq \|D^2 u - D_h^2 u_h\|_{L_2(\Omega)}^2$$

together with (4.18) yields

$$(4.22) \quad h_e^{-3/2} \|\llbracket u_h \rrbracket\|_{L_2(\mathcal{E})} + h_e^{-1/2} \|\llbracket \nabla u_h \rrbracket\|_{L_2(\mathcal{E})} \leq C \|D^2 u - D_h^2 u_h\|_{L_2(\Omega)},$$

concluding the proof.  $\square$

**4.6. Inconsistent nonvariational methods.** We now introduce an inconsistent method for the approximation of (4.1). This is a slight modification of that presented in [DP13] and related to [LP11]. The method we consider is given as follows: Let  $\mathbf{H}(u_h) \in \mathbb{V}^{d \times d}$  be such that for all  $\phi \in \mathbb{V}$

$$(4.23) \quad \int_{\Omega} \mathbf{H}(u_h) \phi = \int_{\Omega} \mathbf{D}_h^2 u_h \phi - \int_{\mathcal{E}} \llbracket \nabla u_h \rrbracket_{\otimes} \llbracket \phi \rrbracket - \llbracket u_h \rrbracket_{\otimes} \llbracket \nabla \phi \rrbracket,$$

then we seek the tuple  $(u_h, \mathbf{H}(u_h))$  such that

$$(4.24) \quad \mathcal{A}_h(u_h, v_h) = \int_{\Omega} f v_h \quad \forall v_h \in \mathbb{V} \cap \mathring{\mathbf{H}}^1(\mathcal{T}),$$

with

$$(4.25) \quad \mathcal{A}_h(u_h, v_h) := \int_{\Omega} \mathbf{A} : \mathbf{H}(u_h) v_h + \int_{\mathcal{E}} \sigma h_e \llbracket \nabla u_h \rrbracket \llbracket \nabla v_h \rrbracket + \int_{\mathcal{E}} \sigma h_e^{-1} \llbracket u_h \rrbracket \cdot \llbracket v_h \rrbracket.$$

**4.7. Theorem** (Stability of  $\mathbf{H}(\cdot)$  [Pry14, Thm 4.11]). *Let  $\mathbf{H}(\cdot)$  be defined as in (4.23), then it is stable in the sense that*

$$(4.26) \quad \|\mathbf{D}_h^2 v_h - \mathbf{H}(v_h)\|_{\mathbf{L}_2(\Omega)^{d \times d}}^2 \leq C \left( \int_{\mathcal{E}} h_e^{-1} \|\llbracket \nabla v_h \rrbracket\|^2 + h_e^{-3} \|\llbracket v_h \rrbracket\|^2 \right).$$

Consequently we have

$$(4.27) \quad \|\mathbf{H}(v_h)\|_{\mathbf{L}_2(\Omega)^{d \times d}}^2 \leq C \|v_h\|_{dG,2}^2.$$

**4.8. Theorem** (Primal a posteriori error upper bound). *Let  $u \in \mathbf{H}^2(\Omega)$  solve (4.2) and  $u_h \in \mathbb{V}$  be the finite element approximation given by (4.24), then*

$$(4.28) \quad \|u - u_h\|_{dG,2} \leq C \left( \sum_{K \in \mathcal{T}} \eta_R^2 + \sum_{e \in \mathcal{E}} \eta_J^2 \right)^{1/2},$$

where

$$(4.29) \quad \eta_R^2 := \|f - \mathbf{A} : \mathbf{H}(u_h)\|_{\mathbf{L}_2(K)}^2$$

$$(4.30) \quad \eta_J^2 := h_e^{-1} \|\llbracket \nabla u_h \rrbracket\|_{\mathbf{L}_2(e)}^2 + h_e^{-3} \|\llbracket u_h \rrbracket\|_{\mathbf{L}_2(e)}^2.$$

**Proof** The proof is similar to that of Theorem 4.4. Using the abstract result of Theorem 2.6 we proceed to bound each  $\mathcal{E}_i$  term. In view of Cauchy–Schwartz we have

$$(4.31) \quad \begin{aligned} \mathcal{E}_1 &= \sup_{v \in \mathbf{L}_2(\Omega), \|v\|_{\mathbf{L}_2(\Omega)} \leq 1} \int_{\Omega} (f - \mathbf{A} : \mathbf{H}(u_h))(v - v_h) \\ &\leq \sup_{v \in \mathbf{L}_2(\Omega), \|v\|_{\mathbf{L}_2(\Omega)} \leq 1} \sum_{K \in \mathcal{T}} \|f - \mathbf{A} : \mathbf{H}(u_h)\|_{\mathbf{L}_2(K)} \|v - v_h\|_{\mathbf{L}_2(K)}. \end{aligned}$$

Choosing  $v_h = 0$  we have

$$(4.32) \quad \mathcal{E}_1 \leq C \sum_{K \in \mathcal{T}} \|f - \mathbf{A} : \mathbf{H}(u_h)\|_{\mathbf{L}_2(K)}.$$

The consistency term  $\mathcal{E}_2 = 0$ . For the second consistency term we have that

$$(4.33) \quad \begin{aligned} \mathcal{E}_3 &= \sup_{v \in \mathbf{L}_2(\Omega), \|v\|_{\mathbf{L}_2(\Omega)} \leq 1} \frac{1}{\gamma} \int_{\Omega} \mathbf{A} : (\mathbf{H}(E^2(u_h)) - \mathbf{D}^2 E^2(u_h)) v \\ &\leq C \|\mathbf{A}\|_{\mathbf{L}_{\infty}(\Omega)} \sum_{K \in \mathcal{T}} \|\mathbf{H}(E^2(u_h)) - \mathbf{D}^2 E^2(u_h)\|_{\mathbf{L}_2(K)} \\ &\leq C \|\mathbf{A}\|_{\mathbf{L}_{\infty}(\Omega)} \sum_{K \in \mathcal{T}} \left( \|\mathbf{H}(E^2(u_h)) - \mathbf{H}(u_h)\|_{\mathbf{L}_2(K)} + \|\mathbf{H}(u_h) - \mathbf{D}_h^2 u_h\|_{\mathbf{L}_2(K)} \right. \\ &\quad \left. + \|\mathbf{D}_h^2 u_h - \mathbf{D}^2 E^2(u_h)\|_{\mathbf{L}_2(K)} \right). \end{aligned}$$

Making use of Lemmata 4.7 and 3.6, we see that

$$(4.34) \quad \sum_{K \in \mathcal{T}} \|\mathbf{H}(E^2(u_h)) - \mathbf{H}(u_h)\|_{L_2(K)} \leq C \|E^2(u_h) - u_h\|_{dG,2} \\ \leq C \left( \sum_{e \in \mathcal{E}} h_e^{-1} \|\llbracket \nabla_h v_h \rrbracket\|_{L_2(e)}^2 + h_e^{-3} \|\llbracket v_h \rrbracket\|_{L_2(e)}^2 \right)^{1/2}.$$

Again from Lemma 4.7 we have that

$$(4.35) \quad \sum_{K \in \mathcal{T}} \|\mathbf{H}(u_h) - D_h^2 u_h\|_{L_2(K)} \leq C \left( \sum_{e \in \mathcal{E}} h_e^{-1} \|\llbracket \nabla_h v_h \rrbracket\|_{L_2(e)}^2 + h_e^{-3} \|\llbracket v_h \rrbracket\|_{L_2(e)}^2 \right)^{1/2}.$$

and Lemma 3.6 gives that

$$(4.36) \quad \sum_{K \in \mathcal{T}} \|D_h^2 u_h - D^2 E^2(u_h)\|_{L_2(K)} \leq C \left( \sum_{e \in \mathcal{E}} h_e^{-1} \|\llbracket \nabla_h v_h \rrbracket\|_{L_2(e)}^2 + h_e^{-3} \|\llbracket v_h \rrbracket\|_{L_2(e)}^2 \right)^{1/2}.$$

Hence we have that

$$(4.37) \quad \mathcal{E}_3 \leq C \left( \sum_{e \in \mathcal{E}} h_e^{-1} \|\llbracket \nabla_h v_h \rrbracket\|_{L_2(e)}^2 + h_e^{-3} \|\llbracket v_h \rrbracket\|_{L_2(e)}^2 \right)^{1/2}.$$

We conclude by noting that from Lemma 3.6

$$(4.38) \quad \mathcal{E}_4 = \|E(u_h) - u_h\|_{dG,2} \leq C \left( \sum_{e \in \widehat{K}} h_e^{-1} \|\llbracket \nabla_h v_h \rrbracket\|_{L_2(e)}^2 + h_e^{-3} \|\llbracket v_h \rrbracket\|_{L_2(e)}^2 \right)^{1/2}.$$

Collecting (4.32), (4.37) and (4.38) yields the desired result.  $\square$

**4.9. Proposition** (Primal a posteriori lower bound). *Using the notation of Theorem 4.8 we have for  $\sigma$  large enough a lower bound of the form:*

$$(4.39) \quad \sum_{K \in \mathcal{T}} \eta_R + \sum_{e \in \mathcal{E}} \eta_J \leq C \left( \|D^2 u - D_h^2 u_h\|_{L_2(\Omega)} \right).$$

**Proof** Notice, as in the consistent case there is no data oscillation, this is since the inconsistency can be linked to the nonconformity via Lemma 4.7. We begin by noting

$$(4.40) \quad \|f - \mathbf{A}:\mathbf{H}(u_h)\|_{L_2(\Omega)} \leq \|\mathbf{A}:(D^2 u - D_h^2 u_h)\|_{L_2(\Omega)} + \|\mathbf{A}:(D_h^2 u_h - \mathbf{H}(u_h))\|_{L_2(\Omega)} \\ \leq C \left( \|D^2 u - D_h^2 u_h\|_{L_2(\Omega)} + \|D_h^2 u_h - \mathbf{H}(u_h)\|_{L_2(\Omega)} \right) \\ \leq C \left( \|D^2 u - D_h^2 u_h\|_{L_2(\Omega)} + h_e^{-1/2} \|\llbracket \nabla u_h \rrbracket\|_{L_2(\mathcal{E})} + h_e^{-3/2} \|\llbracket u_h \rrbracket\|_{L_2(\mathcal{E})} \right),$$

in view of Lemma 4.7. Now, as in the proof of Proposition 4.5 making use of the scheme (4.24) with  $v_h = u_h - E^1(u_h)$  we see

$$(4.41) \quad \sigma h_e^{-1} \|\llbracket u_h \rrbracket\|_{L_2(e)}^2 \leq \sigma h_e^{-1} \|\llbracket \nabla u_h \rrbracket\|_{L_2(\mathcal{E})} \|\llbracket \nabla u_h - \nabla E^1(u_h) \rrbracket\|_{L_2(\mathcal{E})} \\ + \|f - \mathbf{A}:\mathbf{H}(u_h)\|_{L_2(\Omega)} \|u_h - E^1(u_h)\|_{L_2(\Omega)} \\ \leq C \|\llbracket \nabla u_h \rrbracket\|_{L_2(\mathcal{E})} \|\llbracket u_h \rrbracket\|_{L_2(\mathcal{E})} \\ + C h^{1/2} \|\llbracket u_h \rrbracket\|_{L_2(\mathcal{E})} \left( \|D^2 u - D_h^2 u_h\|_{L_2(\Omega)} + h_e^{-1/2} \|\llbracket \nabla u_h \rrbracket\|_{L_2(\mathcal{E})}^2 + h_e^{-3/2} \|\llbracket u_h \rrbracket\|_{L_2(\mathcal{E})}^2 \right).$$

Rearranging the inequality we have

$$(4.42) \quad (\sigma - C) h_e^{-3/2} \|\llbracket u_h \rrbracket\|_{L_2(e)} \leq C h_e^{1/2} \|\llbracket \nabla u_h \rrbracket\|_{L_2(\mathcal{E})} + C \|D^2 u - D_h^2 u_h\|_{L_2(\Omega)}.$$

Choosing  $\sigma$  large enough and using (4.21) concludes the proof.  $\square$

4.10. **Corollary** (A posteriori bounds for the nonvariational problem including data approximation). *Let the conditions of Theorem 4.8 hold. Let  $u_h$  now be the the solution of (4.24) with the following approximations of the bilinear and linear forms*

$$(4.43) \quad \mathcal{A}_h(u_h, v_h) := \sum_{K \in \mathcal{T}} Q_K^{2k-2}(\mathbf{A}:\mathbf{H}(u_h)v_h) + \sum_{e \in \mathcal{E}} Q_e^{2k}(\sigma h_e \llbracket \nabla u_h \rrbracket \llbracket \nabla v_h \rrbracket + \sigma h_e^{-1} \llbracket u_h \rrbracket \cdot \llbracket v_h \rrbracket)$$

$$(4.44) \quad l(v_h) := \sum_{K \in \mathcal{T}} Q_K^{2k-2}(f v_h),$$

then

$$(4.45) \quad \|u - u_h\|_{dG,2} \leq C \left( \sum_{K \in \mathcal{T}} \eta_R^2 + \tilde{\eta}_I^2 + \sum_{e \in \mathcal{E}} \eta_J^2 \right)^{1/2},$$

and for  $\sigma$  large enough

$$(4.46) \quad \sum_{K \in \mathcal{T}} \eta_R + \sum_{e \in \mathcal{E}} \eta_J \leq C \|D^2 u - D_h^2 u_h\|_{L_2(\Omega)} + \sum_{K \in \mathcal{T}} \eta_I$$

where

$$(4.47) \quad \eta_R^2 := \|P_{k-2}f - P_{k-2}(\mathbf{A}:\mathbf{H}(u_h))\|_{L_2(K)}^2$$

$$(4.48) \quad \eta_I^2 := \|(P_{k-2} - \text{Id})(\mathbf{A}:\mathbf{H}(u_h))\|_{L_2(K)}^2 + \|f - P_{k-2}f\|_{L_2(K)}^2$$

$$(4.49) \quad \tilde{\eta}_I^2 := \|(P_{k-2} - \text{Id})(\mathbf{A}:\mathbf{H}(E^2(u_h)))\|_{L_2(K)}^2 + \|f - P_{k-2}f\|_{L_2(K)}^2$$

$$(4.50) \quad \eta_J^2 := h_e^{-1} \|\llbracket \nabla u_h \rrbracket\|_{L_2(e)}^2 + h_e^{-3} \|\llbracket u_h \rrbracket\|_{L_2(e)}^2.$$

**Proof** The proof is the same as that of Theorem 4.8 with the exception that the inconsistency term is slightly more complicated. We have that

$$(4.51) \quad \begin{aligned} \mathcal{E}_3 &= \sup_{v \in L_2(\Omega), \|v\|_{L_2(\Omega)} \leq 1} \frac{1}{\gamma} \int_{\Omega} P_{k-2}(\mathbf{A}:\mathbf{H}(E^2(u_h))) v - \mathbf{A}:D^2 E^2(u_h) v \\ &= \sup_{v \in L_2(\Omega), \|v\|_{L_2(\Omega)} \leq 1} \frac{1}{\gamma} \int_{\Omega} \left[ (P_{k-2}(\mathbf{A}:\mathbf{H}(E^2(u_h))) v - \mathbf{A}:\mathbf{H}(E^2(u_h)) v) \right. \\ &\quad \left. + (\mathbf{A}:\mathbf{H}(E^2(u_h)) v - \mathbf{A}:D^2 E^2(u_h) v) \right]. \end{aligned}$$

The second term in (4.51) can be bounded using the same arguments as in the proof of Theorem 4.8. The other term is controllable by  $\eta_I$  using Cauchy-Schwarz, concluding the proof.  $\square$

4.11. **Remark** (Comparison of the variational and nonvariational estimates). Note the different structures of the data approximation terms appearing in Lemma 3.21 and Corollary 4.10. They are different orders of approximation even though the underlying quadrature approximation is the same. This is because the estimators control the error in different norms due to the different stability frameworks of the underlying PDEs. The data approximation terms are of the same order of accuracy as the other residual terms using a quadrature of degree  $2k - 2$ . Higher order quadrature choice naturally results in the data approximation terms becoming higher order, assuming smooth data.

For the nonvariational scheme the inconsistency is in two distinct parts. The first being the nature of the scheme, which was tackled in Theorem 4.8 and is inherently tied to the nonconforming nature of the scheme. The second is the quadrature approximations. This inconsistency cannot be linked to the nonconformity of the scheme, thus another quantity enters into the estimator.

4.12. **Corollary** ( $H^2$  a posteriori bounds for the Laplacian). *Let  $\mathbf{A} = -\mathbf{I}$  then  $\mathbf{A}:D^2 u = -\Delta u$  then the following a posteriori bound holds for  $u_h$  the approximation given by the IP method under quadrature approximation (3.66)*

$$(4.52) \quad \|u - u_h\|_{dG,2} \leq C \left( \sum_{K \in \mathcal{T}} \eta_R^2 + \eta_I^2 + \sum_{e \in \mathcal{E}} \eta_J^2 \right)^{1/2},$$

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where

$$(4.53) \quad \eta_R^2 := \|P_{k-2}f + \Delta u_h\|_{L_2(K)}^2$$

$$(4.54) \quad \eta_I^2 := \|f - P_{k-2}f\|_{L_2(K)}^2$$

$$(4.55) \quad \eta_J^2 := h_e^{-1} \|\llbracket \nabla u_h \rrbracket\|_{L_2(e)}^2 + h_e^{-3} \|\llbracket u_h \rrbracket\|_{L_2(e)}^2.$$

and

$$(4.56) \quad \sum_{K \in \mathcal{T}} \eta_R + \sum_{e \in \mathcal{E}} \eta_J \leq C \left( \|D^2 u - D_h^2 u_h\|_{L_2(\Omega)} + \|f - P_{k-2}f\|_{L_2(\Omega)} \right).$$

**Proof** The Proof consists of observing that the finite element method (4.24) coincides with the IP method (3.66) (albeit with an additional penalty term for the gradients) whenever  $\mathbf{A}$  is ( $\mathcal{T}$ -wise) constant and applying Theorems 4.8 and 4.9.  $\square$

4.13. **Remark** (Dual a posteriori error control and the formal dual of (4.1)). The formal dual problem is

$$(4.57) \quad \mathcal{A}^*(u, v) = \int_{\Omega} u D^2 \{v \mathbf{A}\}$$

and, to the authors knowledge, there is no general existence result for this class of problem. Under the stringent assumption that the dual problem is well posed, if  $u_h$  solves the nonvariational Galerkin method and  $u$  is a strong solution then

$$(4.58) \quad \|u - u_h\|_{L_2(\Omega)} \leq C \left( \sum_{K \in \mathcal{T}} \eta_{2,R}^2 + \sum_{e \in \mathcal{E}} \eta_{2,J}^2 \right)^{1/2},$$

where

$$\begin{aligned} \eta_{2,R}^2 &:= h_K^4 \|f - \mathbf{A} : \mathbf{H}(u_h)\|_{L_2(K)}^2 \\ \eta_{2,J}^2 &:= h_e^3 \|\llbracket \nabla u_h \rrbracket\|_{L_2(e)}^2 + h_e \|\llbracket u_h \rrbracket\|_{L_2(e)}^2 \end{aligned}$$

This result is included as, from a practical point of view, the estimators seem to be reliable and efficient.

## 5. NUMERICAL EXPERIMENTS

In this section we present some numerical results testing both the asymptotic behaviour of the estimators derived in previous sections as well as its effectiveness as an indicator for adaptivity.

5.1. **Inconsistency terms from §3.** We begin this section with an adaptive numerical test problem for the estimator given in Lemma 3.21. We take the diffusion coefficient to be

$$(5.1) \quad \mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & \arctan \left( 1000 \left( \left( x + \frac{1}{2} \right)^2 + \left( y - \frac{1}{2} \right)^2 - 1 \right) \right) + 2 \end{bmatrix}$$

and  $\Omega = [0, 1]^2 - [\frac{1}{2}, 1] \times [0, \frac{1}{2}]$  to be the standard L-shaped domain. We use a maximum strategy adaptive algorithm given in Algorithm 1 [SS05] and the estimator given in Lemma 3.21. We use  $k = 2$  and take the quadrature degree to be order 2. Figure 1 shows the numerical solution at the final level of refinement and Figure 5 shows the mesh generated and estimator contributions at various levels of refinement. Note the solution will be singular at the nonconvex corner of the domain. In addition the problem data is chosen such that  $\mathbf{A}$  has a very large gradient over a large portion of the domain which induces a further singularity. Both seem to be captured correctly using this algorithm.

---

**Algorithm 1** Maximum strategy type adaptive algorithm incorporating inconsistencies.

---

Given constants  $C_1, C_2, C_3$  such that  $C_1 + C_2 + C_3 = 1$ , tol and  $\xi, \kappa$ ;

Set  $N = 0$ ;

**while**  $C_1 H_R + C_2 H_I + C_3 H_J > \text{tol}$  AND  $N \leq \text{Max Iterations}$  **do**

**for all**  $K \in \mathcal{T}$  **do**

    Compute  $\eta_R, \eta_J$  and  $\eta_I$ ;

$H_R = H_R + \eta_R^2$ ;

$H_I = H_I + \eta_I^2$ ;

**for all**  $e \in K$  **do**

$H_J = H_J + \frac{1}{2} \eta_{J,e}^2$ ;

**end for**

    Set  $\eta_K = \eta_R + \frac{1}{2} \eta_J + \eta_I$ ;

**if**  $\eta_K \geq \xi \max_{L \in \mathcal{T}} (\eta_L)$  **then**

      Mark element  $K$  for refinement;

**end if**

**if**  $\eta_K \leq \kappa \max_{L \in \mathcal{T}} (\eta_L)$  **then**

      Mark element  $K$  for coarsening;

**end if**

**end for**

$H_R = \sqrt{H_R}, H_J = \sqrt{H_J}, H_I = \sqrt{H_I}$ ;

  Perform any mesh change;

$N = N+1$ ;

**end while**

---

FIGURE 1. The adaptive approximation to the solution of the diffusion problem with problem data given in §5.1.

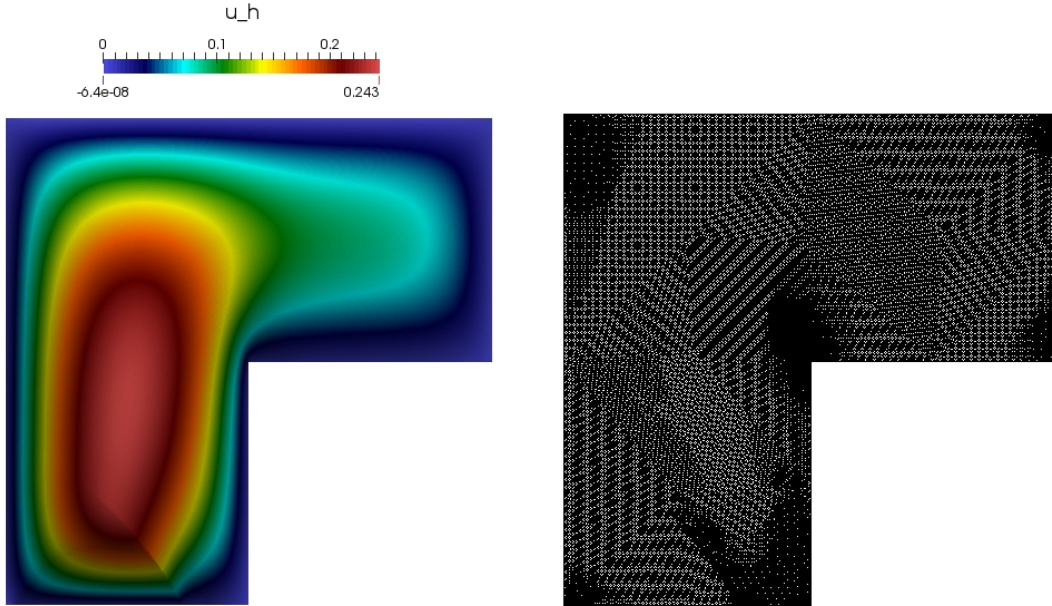
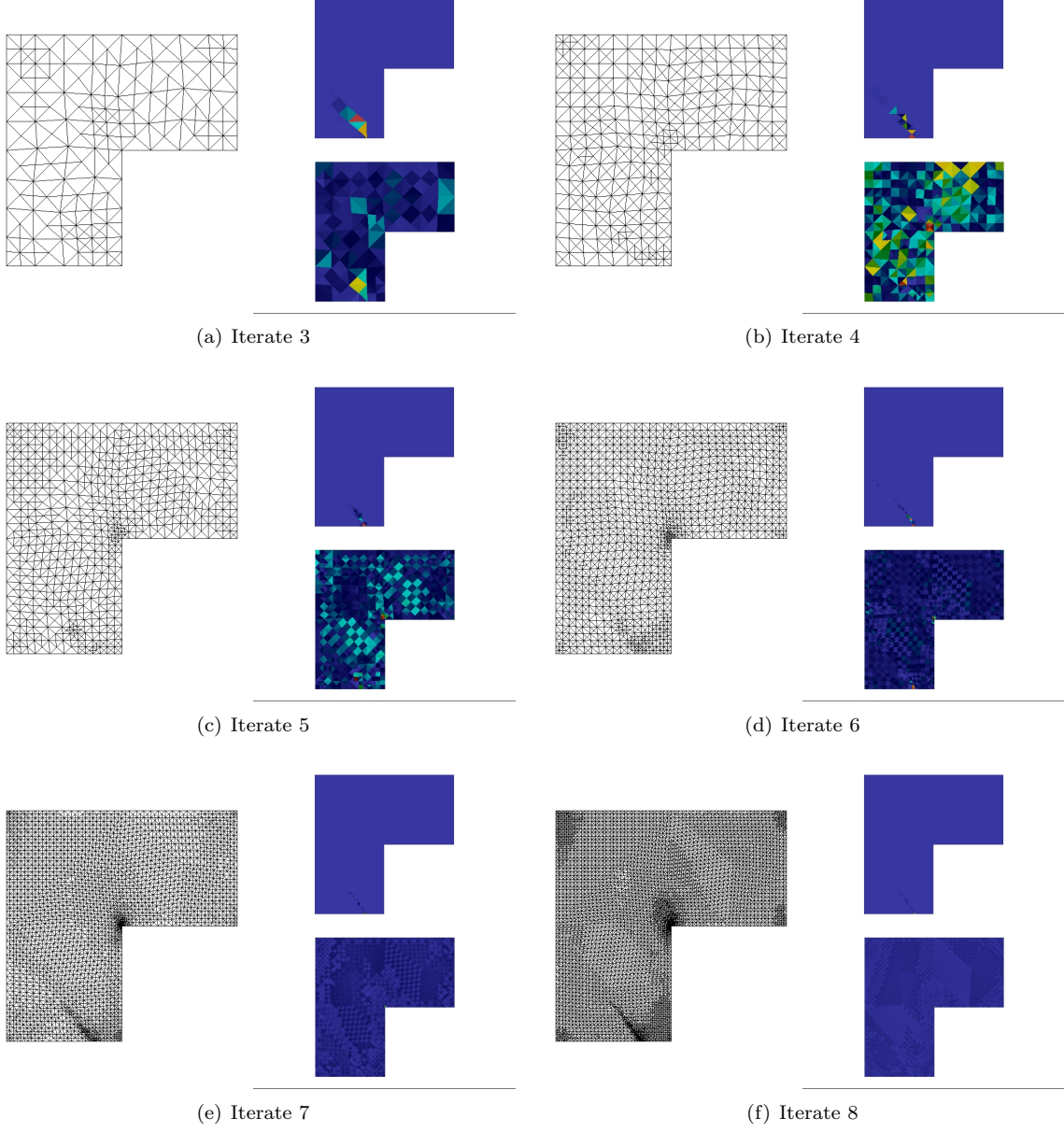




FIGURE 2. Testing the estimator given in Lemma 3.21 as a driver for adaptivity. The problem data is described in §5.1. We consider various iterates of the adaptive procedure looking at the mesh and the estimator components. The top estimator is the inconsistency terms and the bottom the interior and jump residual. Notice the inconsistency term is extremely localised to where the diffusion coefficient has large gradient and becomes comparatively negligible once the mesh is sufficiently resolved in that region.



**5.2. Inconsistent nonvariational approximations from §4.** We are studying both the primal and dual estimators derived in §4 for nonconforming, inconsistent schemes approximating nonvariational problems. We examine the asymptotic behaviour of the estimators from Theorems 4.8 and 4.13 and also their effectiveness as drivers for adaptive algorithms.

*Test 1 : Asymptotic behaviour.* With  $d = 2$  if  $\mathbf{A}:\mathbf{D}^2u$  is a uniformly elliptic operator then the Campanato condition for strong solutions is satisfied [Tar04] and Assumption 2.1 is satisfied. We fix  $\Omega = [\frac{1}{2}, \frac{1}{2}]^2$  use the problem data

$$(5.2) \quad \mathbf{A} = \begin{bmatrix} 2 & 0 \\ 0 & \sin(2\pi x) \sin(2\pi y) + 2. \end{bmatrix}$$

We have chosen  $\mathbf{A}$  in this fashion such that it varies slowly over the domain and the inconsistency term arising from quadrature approximation is negligible. We choose  $f$  such that the exact solution is given by either

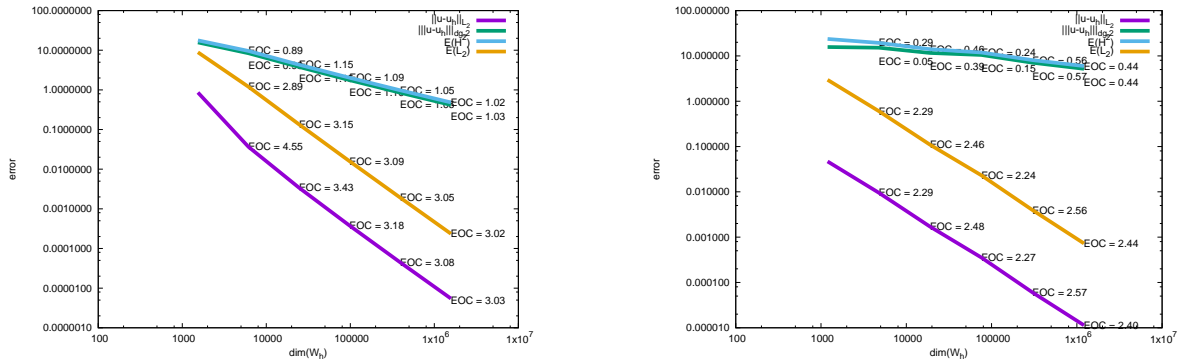
$$(5.3) \quad u(\mathbf{x}) = \sin(2\pi x) \sin(2\pi y) \in C^\infty(\Omega)$$

or

$$(5.4) \quad u(\mathbf{x}) = \begin{cases} \frac{1}{4} \left( \cos\left(8\pi \left|\mathbf{x} - \frac{1}{2}\right|^2\right) + 1 \right) & \text{if } \left|\mathbf{x} - \frac{1}{2}\right|^2 \leq \frac{1}{8} \\ 0 & \text{else} \end{cases} \in H^2(\Omega) / H^3(\Omega).$$

In Figure 3 we summarise the results of this test and demonstrate that computationally the asymptotic convergence rate of the estimator is the same as that of the error as predicted by the theory given in Theorems 4.8 and 4.9.

FIGURE 3. Testing the asymptotic behaviour of the estimator given in Theorem 4.8. The problem data is described in §5.2 - Test 1.



(a) When  $u \in C^\infty(\Omega)$  is given by (5.3)

(b) When  $u \in H^2(\Omega) / H^3(\Omega)$  is given by (5.4)

*Test 2 : Adaptive algorithms.* We fix  $\Omega = [\frac{1}{2}, \frac{1}{2}]^2$ , as in Test 1, and choose  $\mathbf{A}$  as in (5.2). We choose  $f$  such that the exact solution is given in (5.4). Note this function is  $H^2(\Omega) / H^3(\Omega)$ . We use the adaptive algorithm given in Algorithm 1 and, in Figure 4, examine the convergence of said algorithm. The rates are increased from those given in Figure 3 (approximately  $O(N^{-1/4})$ ) to  $O(N^{-1/2})$  when measured in the  $H^2(\Omega)$  norm.

*Test 3 : Quadrature effects, smooth solution.* We fix  $\Omega = [\frac{1}{2}, \frac{1}{2}]^2$ , as in Test 1 and 2, and choose

$$(5.5) \quad \mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & a_{11}(\mathbf{x}) \end{bmatrix}$$

where

$$(5.6) \quad a_{11}(\mathbf{x}) = \begin{cases} 10 \sin(100\pi x) \sin(100\pi y) + 11 & \text{if } x, y \geq 0 \\ 11 & \text{otherwise.} \end{cases}$$

Notice the diffusion coefficient oscillates much more heavily in the upper right quadrant of  $\Omega$  than that of the previous tests. The is to force the inconsistency term arising from quadrature to have a larger impact. We choose  $f$  such that the exact solution is smooth and given in (5.3). We use the adaptive algorithm given in Algorithm 1 and, in Figure 4, examine the behaviour of the inconsistency term compared to the rest of the estimator. The increase in size and oscillations of the diffusion term force the inconsistency term in the estimator to dominate at the initial coarse mesh scale which results in a denser mesh in the upper right quadrant of  $\Omega$ .

*Test 4 : Quadrature effects, nonsmooth solution.* This test is the same as Test 4, with the exception that the solution is given by (5.4). Figure 6 summaries the results.

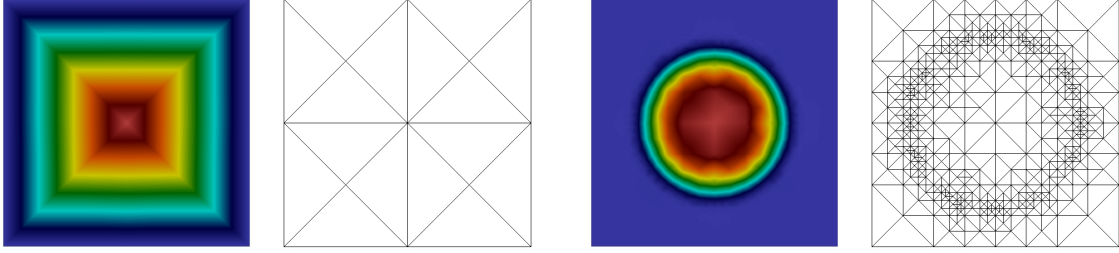
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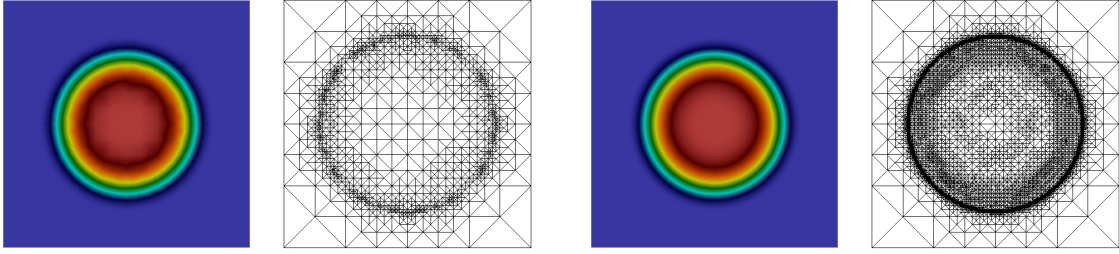
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FIGURE 4. Testing the estimator given in Theorem 4.8 as a driver for adaptivity. The problem data is described in §5.2 - Test 2 such that the solution is given by (5.4). We consider various iterates of the adaptive procedure governed by Algorithm 1 looking at the mesh generated. Notice the mesh is refined around where the solution is nonsmooth and the convergence rate of the adaptive algorithm is quicker than the uniform counterpart from Test 1.



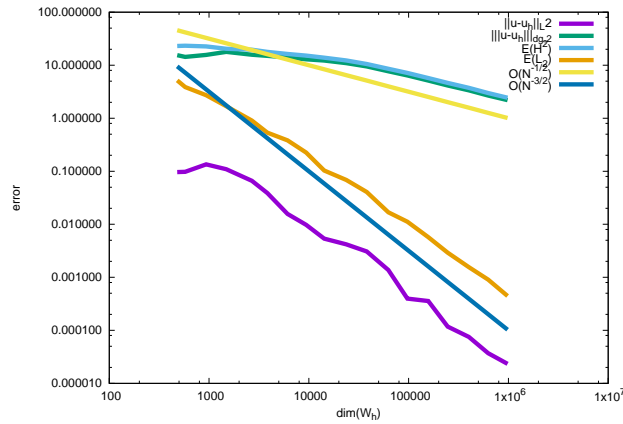
(a) Iterate 1.

(b) Iterate 11.



(c) Iterate 13.

(d) Iterate 18.



(e) Convergence rates for the adaptive algorithm.

FIGURE 5. Testing the estimator given in Theorem 4.8 as a driver for adaptivity. The problem data is described in §5.2 - Test 3. In this case the solution is smooth and the diffusion coefficient oscillates heavily in the upper right quadrant. We consider various iterates of the adaptive procedure looking at the mesh and the estimator components. The top estimator is the inconsistency terms and the bottom the interior and jump residual. Notice after the mesh is sufficiently resolved where the oscillations are heavy, the inconsistency term becomes comparatively negligible.

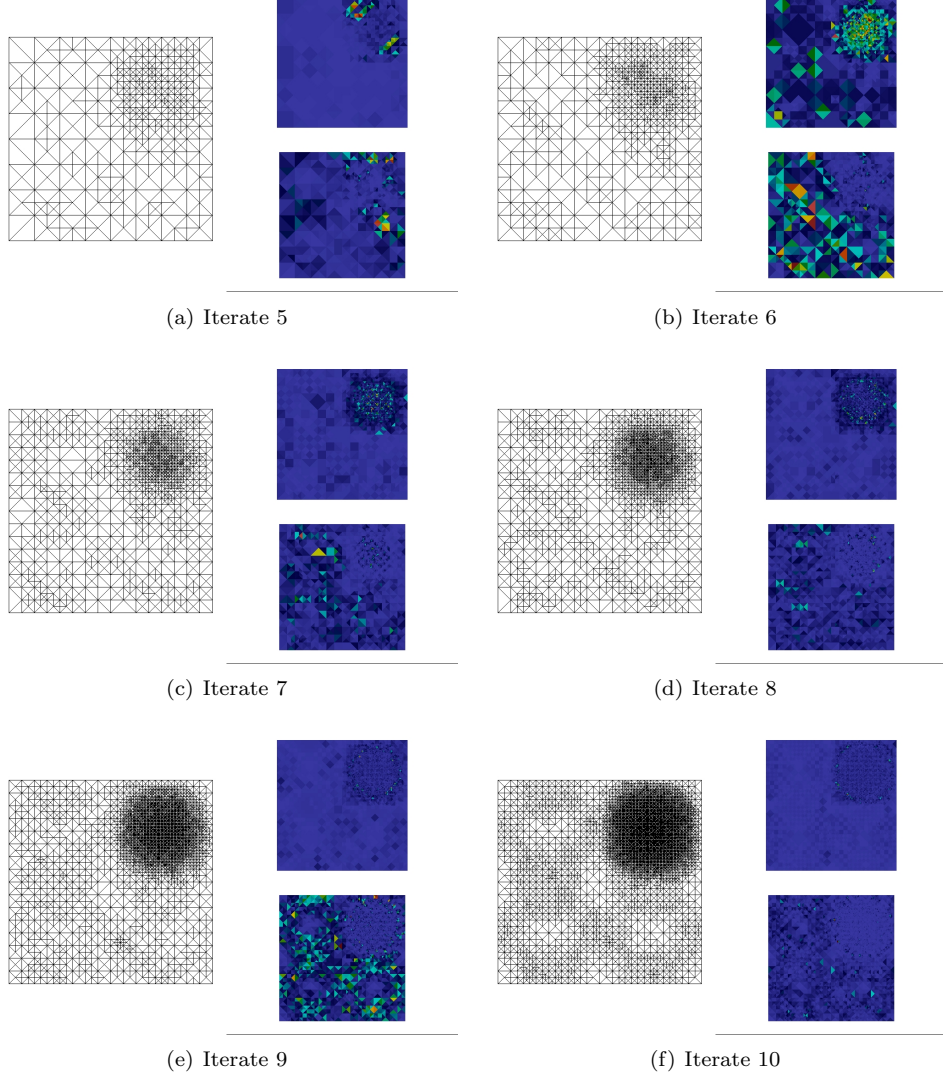


FIGURE 6. Testing the estimator given in Theorem 4.8 as a driver for adaptivity. The problem data is described in §5.2 - Test 4. In this case the solution is not smooth and the diffusion coefficient oscillates heavily in the upper right quadrant. We consider various iterates of the adaptive procedure looking at the mesh and the estimator components. The top estimator is the inconsistency terms and the bottom the interior and jump residual. Notice the mesh is very resolved around where the solution is not smooth and also where the oscillations in the problem data are heavy.

